J. ENVIRONMENTAL SYSTEMS, Vol. 11(2), 1981-82

ANALYTICAL PREDICTION OF VERTICAL TEMPERATURE DISTRIBUTION IN LARGE WATER BODIES*

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ABSTRACT

An analytical solution is presented which allows one to calculate the vertical temperature distribution in deep water bodies such as lakes and reservoirs where inflows and outflows are negligible. The solution is based on a linearization of the surface heat exchange term. The details of the linearization and the rationale for its use are presented. The solution is valid for both variable and constant meteorological conditions. Comparisons with both field observations and laboratory data are shown to verify the model.

INTRODUCTION

The prediction of vertical temperature distribution in lakes or large water bodies has received attention in the literature throughout the years. It is an important parameter in the analysis of these systems. Dissolved oxygen content, suspended solids, dissolved mineral content, and biological activity are all functions of temperature. Therefore, accurate prediction of the temperature distribution in a water body will allow better understanding of the ecology of the system.

Dake and Harleman developed a one-dimensional model for vertical temperature distribution in a deep stagnant water body [1]. They developed analytical solutions for three cases by specifying mathematical functions for the net insolation and the surface heat losses. No detailed physical rationale for

* This work was supported by a grant from the Wyoming Mining and Mineral Resources and Research Institute.

doi: 10.2190/9YHL-T4UG-5GW2-J9W7
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these cases was given. Comparison of the models with both field and laboratory experiments was generally good. The model also accounts for bouyant mixing by generating a surface mixed layer when the system is unstable. This approach neglects any advective heat transfer so inflows and outflows are negligible.

Snider and Viskanta also developed a one-dimensional model for the vertical temperature distribution in stagnant water bodies [2]. They present an analytical solution based on a linearization of the net heat exchange at the air-water interface. This linearization is based on the difference between the surface water temperature and the ambient air temperature. No details of the linearization were given. A three term exponential decay equation was used for the volumetric rate of absorption of radiant energy by the water. The model agreement with laboratory data was very good.

The objective of this study is to present an analytical solution for the vertical temperature distribution in large stagnant water bodies which takes into account constant or variable surface energy conditions. The model contains no fitted parameters. The solution is based on a linearization of the surface heat exchange term. The linearization is described in detail and the rationale for its use as compared to other methods is described. This method has previously been shown to predict temperature variations in rivers [3]. Comparison with published data is used to verify the model.

SURFACE HEAT EXCHANGE

The major contribution to the vertical temperature distribution is the net surface heat exchange. The net exchange (H) can be written as

$$H = H_I - H_{BR} - H_C - H_E - H_A \tag{1}$$

Here, H_I = net incoming short and long-wave radiation actually absorbed at the surface, H_C = conduction heat loss, H_E = evaporative heat loss, and H_A = advected heat loss.

$$H_I = \beta I_o \tag{2}$$

where I_o = net incoming short- and long-wave radiation which is absorbed by the water body and β = fraction absorbed at the surface.

$$H_{BR} = \epsilon\sigma(T_s + \Delta)^4 \tag{3}$$

where ϵ = emissivity of water, σ = Stefan-Boltzmann constant, T_s = temperature of the water surface, and Δ = scale factor to shift temperature to an absolute scale.

$$H_E = \rho U\lambda(e_T - e_a) \tag{4}$$

where ρ = density of water, U = wind speed function, λ = latent heat of vaporization of water, e_T = saturated vapor pressure of water at the water surface temperature, and e_a = actual vapor pressure of water in air.

$$H_C = C_1 \frac{(T_s - T_a)}{(e_T - e_a)} H_E \tag{5}$$

where T_a = dry-bulb air temperature and

$$C_1 = 0.61 \frac{P}{P_{atm}} \tag{6}$$

where P = actual atmospheric pressure above the water surface and P_{atm} = 1 atmosphere pressure in the same units as P. H_A is assumed small and is neglected. A more detailed description of the above terms is given elsewhere [4].

Any analytical solution to transient temperature distributions is hindered by the non-linearity of the back radiation and vapor pressure terms. To overcome this obstacle, a linearization of the N_{BR}, H_C, and H_E terms is undertaken. These terms are expanded in a Taylor series expansion about a base temperature T_b and terms larger than first order are neglected. It has been shown that this is the optimum linearization method when the base temperature chosen is the initial surface water temperature [5].

$$H \approx \beta I_o + \gamma - \delta T_s \tag{7}$$

$$\gamma = -\epsilon\sigma(T_b + \Delta)^4 - \rho U\lambda[\,(e_{T_b} - e_a) + C_1(T_b - T_a)\,] + 4\epsilon\sigma T_b(T_b + \Delta)^3 + \rho U\lambda[\,(\frac{\partial e_T}{\partial T})_{T_b} + C_1\,] \tag{8}$$

$$\delta = 4\epsilon\sigma(T_b + \Delta)^3 + \rho U\lambda[\,(\frac{\partial e_T}{\partial T})_{T_b} + C_1\,]\,. \tag{9}$$

SOLUTION OF GOVERNING DIFFERENTIAL EQUATION

The differential equation which describes the vertical temperature distribution in a large water body is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} + \frac{I_o a}{\partial C_p}(1 - \beta)e^{-az} \tag{10}$$

where z = vertical distance in water from the surface, α = thermal diffusivity of water, t = time, C_p = isobaric heat capacity of water, and a = extinction coefficient for radiation penetration in water.

The initial condition is

$$@ t = 0 \qquad T = T_o \tag{11}$$

This corresponds to the physical situation in spring where the entire water body is at a uniform temperature.

The boundary conditions are

$$z \to \infty \qquad T \to \text{finite} \tag{12}$$

$$z = 0 \qquad \beta I_o + \gamma - \delta T = -\rho C_p \alpha \frac{\partial T}{\partial z} \tag{13}$$

Equation (12) states that the solution must exist for the entire depth of the lake and equation (13) states that the net heat exchange at the surface equals the heat conducted into the water. Here I_o is assumed constant.

The solution to equation (10) under these conditions is

$$\begin{aligned} T = {} & \frac{D}{C} + \frac{CT_o - D}{C}\left[\operatorname{erf}\left(\frac{z}{2\sqrt{\alpha t}}\right) + e^{cz + c^2 \alpha t} \operatorname{erfc}\left(\frac{z}{2\sqrt{\alpha t}} + c\sqrt{\alpha t}\right)\right] \\ & + \frac{A}{\alpha a^2}\left[2a\sqrt{\alpha t}\ \operatorname{ierfc}\left(\frac{z}{2\sqrt{\alpha t}}\right) - e^{-az} + \frac{1}{2} e^{a^2 \alpha t - az}\right. \\ & \left.\operatorname{erfc}\left(a\sqrt{\alpha t} - \frac{z}{2\sqrt{\alpha t}}\right) + \frac{1}{2} e^{a^2 \alpha t + az} \operatorname{erfc}\left(a\sqrt{\alpha t} + \frac{z}{2\sqrt{\alpha t}}\right)\right] \end{aligned} \tag{14}$$

$$A = \frac{I_o a}{\rho C_p}(1 - \beta) \tag{15}$$

$$C = \frac{\delta}{\rho C_p \alpha} \tag{16}$$

$$D = \frac{\beta I_o + \gamma}{\rho C_p \alpha} \tag{17}$$

If C = 0, which corresponds to constant meteorological conditions, the solution becomes

$$\begin{aligned} T = {} & T_o + 2D\left[\frac{\sqrt{\alpha t}}{\pi} e^{-\frac{z^2}{4\alpha t}} - \frac{z}{2} \operatorname{erfc}\left(\frac{z}{2\sqrt{\alpha t}}\right)\right] \\ & + \frac{A}{\alpha a^2}\left[2a\sqrt{\alpha t}\ \operatorname{ierfc}\left(\frac{z}{2\sqrt{\alpha t}}\right) - e^{-az} + \frac{1}{2} e^{a^2 \alpha t - az}\right. \\ & \left.\operatorname{erfc}\left(a\sqrt{\alpha t} - \frac{z}{2\sqrt{\alpha t}}\right) + \frac{1}{2} e^{a^2 \alpha t + az} \operatorname{erfc}\left(a\sqrt{\alpha t} + \frac{z}{2\sqrt{\alpha t}}\right)\right] \end{aligned} \tag{18}$$

Dake and Harleman noted that at some stage in the yearly cycle, the temperature will increase with depth to some maximum temperature and then decrease as you proceed further in depth [1]. The resulting density distribution in this surface region is unstable and vertical mixing will take place to some finite depth causing a surface mixed layer. To calculate this depth (h), an energy balance yields (1)

$$\int_{o}^{h} (T - T_m)dz = 0 \tag{19}$$

$$@z = h \qquad T = T_m \tag{20}$$

where T_m = temperature of the surface mixed layer.

Equations (19) and (20) allow one to calculate h and T_m whenever this unstable situation arises.

COMPARISON WITH EXPERIMENTAL DATA

To test the model, it was compared with both field observations and laboratory data previously reported in the literature [1, 6]. Goldman and Carter measured vertical temperatures in Lake Tahoe for a 120-day period [6]. Dake and Harleman reported fairly constant meteorological and incoming radiation conditions during the time period of the study [1]. For this data,

$a = 0.05\ m^{-1}$, $\beta = 0.40$, $\delta = 0$, and $\gamma = 1.45 \times 10^2 \frac{W}{m^2}$. Since $\delta = 0$, $C = 0$, equations

(18), (19), and (20) were used. As shown in Figure 1, agreement with field observations is very good and in almost every case is much less than 1°C. The solution is sensitive to the value of a used and a more accurate description of a would result in even better agreement.

Dake and Harleman also report laboratory data [1]. Under these conditions, infrared lamps were used to supply a constant incoming radiation and the vertical temperature distribution was measured for a four-hour period. Figure 2 shows model predictions and experimental results. For this case, $a = 1.0\ m^{-1}$,

$\beta = 0.75$, $\delta = 19.46 \frac{W}{m^2 \cdot k}$, and $\gamma = 509.47 \frac{W}{m^2}$. γ and δ were determined for

this case by generating the surface heat losses according to Dake and Harleman's formula (see their equation (24)), and then plotting this result versus surface water temperature. In this case, the surface was stable and equation (14) was sufficient. As seen from Figure 2, the results are again generally very good. The worst case occurs after twenty-four hours. This again could be attributed to the sensitivity of the solution upon the value of a chosen.

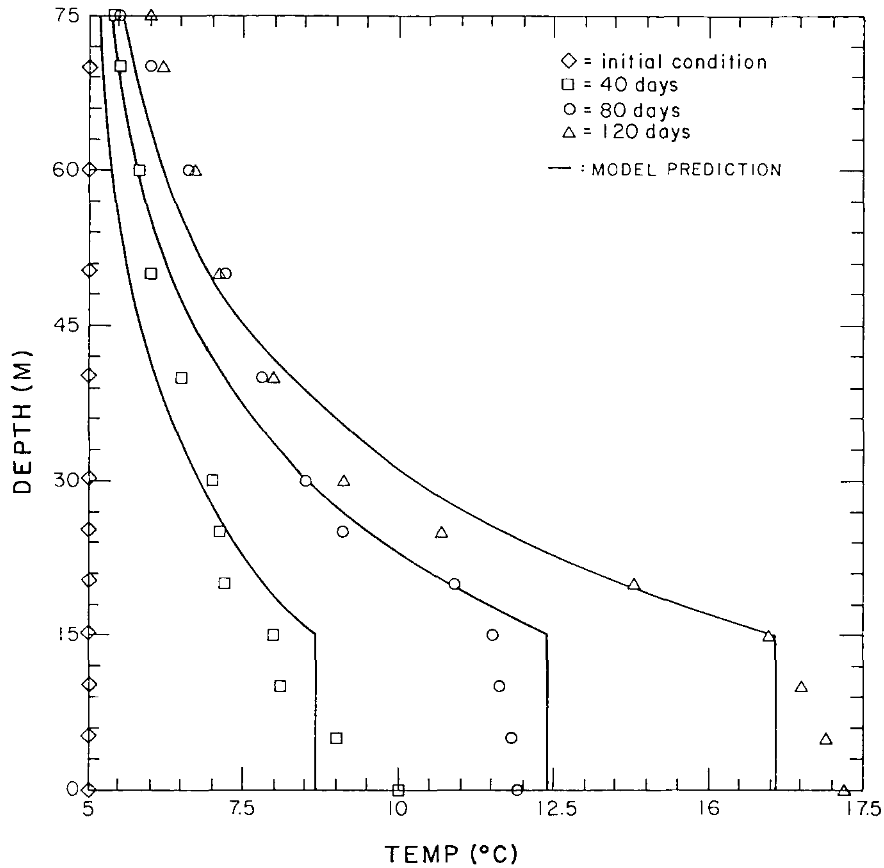

Figure 1. Model prediction and experimental results for Lake Tahoe.

CONCLUSIONS

An analytical equation is presented which describes the vertical temperature distribution in large water bodies with negligible advection. The solution contains no fitted parameters. The equations presented are based on a linearization of the surface heat losses. The linearization is shown in detail and the rationale for its use is explained.

The solutions presented allow one to accurately determine vertical temperature distributions from a knowledge of the optical characteristics of the water, the incoming radiation intensity, and the meteorological conditions at the surface.

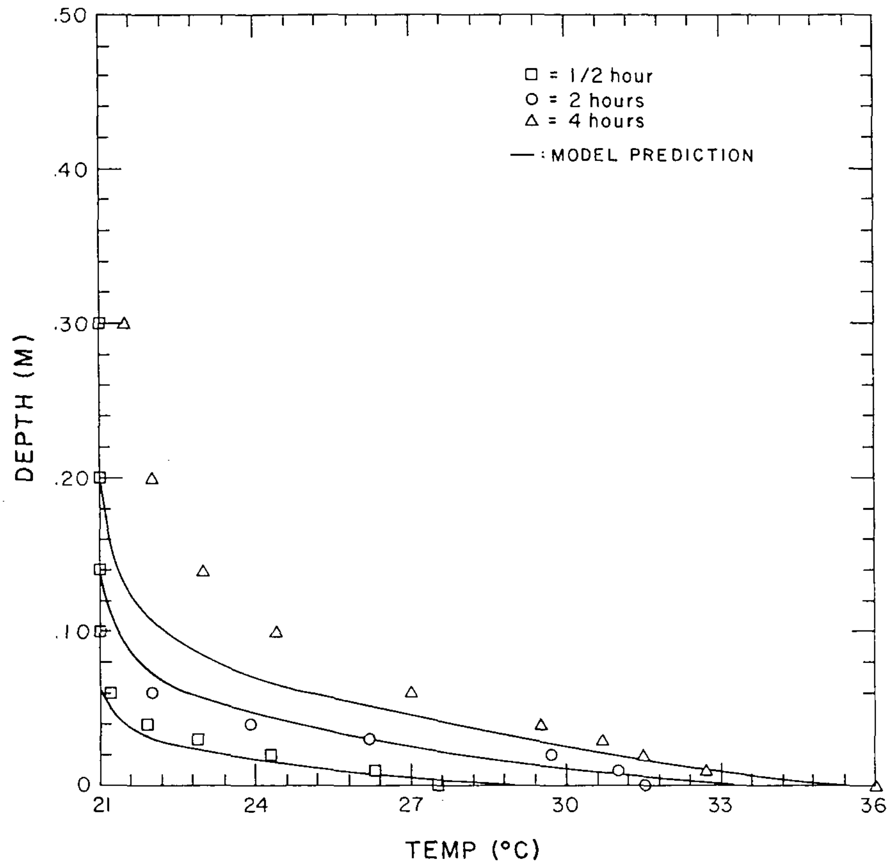

Figure 2. Model prediction and experimental results for a laboratory system.

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