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STUMPAGE PRICE UNCERTAINTY AND THE OPTIMAL ROTATION OF A FOREST: AN APPLICATION OF SANDMO MODEL

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ABSTRACT

The Faustmann model has played a key role in the determination of the optimal forest rotation. Faustmann developed a simple and deterministic competitive economic model, the objective of which was to maximize the present value of perpetual returns to the fixed factor, a unit of timberland [1]. The optimal rotation problem thus viewed is a timber management problem abstracting from any environment of uncertainty. This article considers an alternative model formulation that treats a forest resource operated under conditions of stumpage price uncertainty and forest owners with risk aversion. A modified Faustmann-type rule under conditions of stumpage price uncertainty is developed based on the theory of a competitive firm under price uncertainty developed by Sandmo [2]. The Sandmo model is then used to investigate the effects of an increasing risk on the optimal rotation age.

THE MODEL

In the initial analysis presented, the forest consists of a single homogeneous tree population distributed uniformly and initially grown on a bare plot of land. The forest manager is assumed to be operating in a perfectly competitive market and to have perfect knowledge of the level of the tree population (tree stock) and the regeneration and harvesting costs. Like Faustmann, we abstract from the nontimber benefits flowing from a standing forest.

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Assuming a competitive market, let G(t) denote the stumpage value (net of harvesting cost) in a forest of age t. Once the optimal harvest age is determined, harvesting costs are likewise determined. Hence, G(t) can be considered as net stumpage value.

It is assumed that $G(t) \gtrless 0$. Initially G(t) increases at an increasing rate, then at a decreasing rate, reaches a maximum, falls, and then probably levels off at a steady state. Since the forest stand is regenerated naturally on initially base land at time t = 0, regeneration cost is ignored.

It is assumed that the forest manager considers only the stumpage price p stochastic with a subjective probability density function $\phi(p)$ and an expected price, $E[p] = \overline{p}$, where E is the expectation operator. Furthermore, it is assumed that the planting decision when the production process starts must be taken *ex ante*, i.e., before the stumpage price is known, and only on the basis of the knowledge of the price summarized in the density function. To facilitate comparison with the deterministic model, the stochastic price can be subsumed in stochastic stumpage value. If G(t) is the stumpage value (net of harvesting cost) of a forest of age t with a stochastic price, then G(t) is stochastic with a subjective density function f[G(t)] and an expected stumpage value $E[G(t)] = \overline{G}(t)$.

The forest manager is assumed to choose a rotation cycle to maximize the expected utility of discounted value of all net returns from the forest resource calculated over the infinite chain of renewal cycles. The net return from a single rotation is given by

$$V_1(T) = G(T)e^{-rT} \tag{1}$$

where $r > 0$ is the discount rate, T is the age of the trees for each rotation cycle, and G(T) is a random variable of stumpage value.

Given that all rotations are alike, the net return from all future rotations is given by

$$V(T) = \frac{1}{1 - e^{-rT}} G(T)e^{-rT} = V_1(T)/1 - e^{-rT} . \tag{2}$$

The approach adopted here is to describe the rotation problem in terms of the classic Von Neumann-Morgenstern theory of individual decision making under uncertainty. Uncertainty in stumpage price results in a V that is stochastic. Hence, the manager must select the best of the available probability distributions for V, which are called random prospects. If we assume that the manager's behavior in solving this problem conforms to the Von Neumann-Morgenstern axioms [3], then it can be inferred that the preference ordering for various random prospects can be represented by a utility function U[V(t)] and that the best prospect is found by maximizing the expected value of utility.

For a forest manager with a planning horizon running through one harvest cycle from the time t = 0 through t = T, the objective function to be maximized with respect to T can be written as

$$W_1(T) = E\left\{ U[V_1(T)] \right\}. \tag{3}$$

When the planning horizon is extended to an infinite sequence of identical harvest cycles, the objective function to be maximized turns out to be

$$W(T) = E\left\{ U[V_1(T)/(1 - e^{-rt})] \right\}. \tag{4}$$

The forest manager's attitude towards risk in resource return is represented by the form of the U[V(T)]. Strict concavity in the utility function implies risk aversion. The choice of the particular form is based on its risk characteristics in terms of the measures of risk aversion developed by Arrow [4] and Pratt [5]. In the analysis here, utility is represented by a concave, continuous, and twice differentiable function of discounted net returns, U[V(t)], where

$$U'[V(T)] > 0, \quad U''[V(T)] < 0, \tag{5}$$

so that the forest manager is assumed to be risk averse.

For clarity and convenience of exposition, the analysis runs in terms of two cases: the Fisherian one-cycle case and the Faustmann many-cycle case, remembering of course that the Faustmann formulation is the only correct one [3].

Fisherian One-Cycle Solution

For a one-cycle time horizon, the expected utility of the discounted net return from a forest of age T is:

$$E\left\{ U[V_1(T)] \right\} = \int U[e^{-rT}G(T)]\, f[G(T)]\, dG(T), \tag{6}$$

where the first integration is over the range of G(T). Alternatively stated,

$$E\left\{ U[V_1(T)] \right\} = E\left\{ U[e^{-rT}G(T)] \right\}. \tag{7}$$

Differentiating equation (7) with respect to T, the necessary condition for an optimum is

$$E\left\{ U'[V_1(T)]\,[G'(T) - rG(T)] \right\} = 0, \tag{8}$$

and the sufficient condition for an optimum is

$$\begin{aligned} D = E\Big\{ & U''[V_1(T)]\,[G'(T) - rG(T)]^2 e^{-rT} + \\ & U'[V_1(T)]\,[G''(T) - rG'(T)] - r[G'(T) - rG(T)] \Big\} < 0. \end{aligned} \tag{9}$$

If $[G''(T) - rG'(T)] < r[G'(T) - rG(T)]$, then $D < 0$ is satisfied.

It is assumed that equations (8) and (9) determine a nonzero, finite, and unique solution T, say T*, to the present maximization problem. Under certainty, the solution T is characterized by the equality between the net gain from marginal time (G′(T)) and the opportunity cost of marginal time (rG(T)).

To allow for the comparison between the competitive optimal rotation under conditions of certainty and uncertainty, following Sandmo [2], the problem is posed as follows: What is the optimal rotation time under uncertainty compared to the situation where the stumpage price is known to be equal to the expected value of the original distribution. The latter time is referred to as the deterministic time.

Now the first-order condition equation (8) can be rewritten as

$$E\left\{U'[V_1(T)]G'(T)\right. = E\left\{U'[V_1(t)]rG(T)\right\}. \tag{10}$$

Subtracting $E\left\{U'[V_1(T)]E[rG(T)]\right\}$ from and adding $E\left\{U'[V_1(T)]E[G'(T)]\right\}$ to both sides of equation (10), and remembering that $E[rG(T)] = r\bar{G}(T)$ and $E[G'(T)] = \bar{G}'(T)$, we have

$$E\left\{U'[V_1(T)][\bar{G}'(T) - r\bar{G}(T)]\right\} = E\left\{U'[V_1(T)]\right.$$
$$\left.[rG(T) - r\bar{G}(T) + \bar{G}'(T) - G'(T)]\right\}. \tag{11}$$

Since $E[V_1(T)] = E[G(T)]e^{-rT}$ (from the definition of $V_1(T)$), we have $V_1(T) = E[V_1(T)] + [G(T) - \bar{G}(T)]e^{-rT}$. Given the concavity of U, it then follows that

$$U'[V_1(T)] < U'\left\{E[V_1(T)]\right\}, \tag{12}$$

if $G(T) > \bar{G}(T)$. Then,

$$U'[V_1(T)][rG(T) - r\bar{G}(T) + \bar{G}'(T) - G'(T)] <$$
$$U'\left\{E[V_1(T)\right\}rG(T) - r\bar{G}(T) + \bar{G}'(T) - G'(T)]. \tag{13}$$

This inequality holds for all G and G′ [2]. Taking expectations on both sides of equation (13) and noting that $U'\left\{E[V_1(T)]\right\}$ is a given number,

$$E\left\{U'[V_1(T)][rG(T) - r\bar{G}(T) + \bar{G}'(T) - G'(T)]\right\} <$$
$$U'\left\{E[V_1(T)]\right\}E[rG(T) - r\bar{G}(T) + \bar{G}'(T) - G'(T)]. \tag{14}$$

But, here the right-hand side is equal to zero by definition, and, therefore, the left-hand side is negative. Consequently, the left-hand side of equation (11) is also negative, i.e.,

$$E\left\{U'[V_1(T)]\right\}[\bar{G}'(T) - r\bar{G}(T)] < 0. \tag{15}$$

Since marginal utility $(U'[V_1(T)])$ is positive, this implies that

$$\bar{G}'(T) < r\bar{G}(T), \tag{16}$$

or

$$\bar{G}'(t)/\bar{G}(T) < r. \tag{17}$$

Inequality equation (16) shows that the expected utility maximizing rotation time T is characterized by the expected net return of marginal time, $\bar{G}'(T)$, being less than the expected opportunity cost of marginal time $r\bar{G}(T)$. Or, in other words, in terms of equation (17), it means at the optimal rotation age the expected growth rate of stumpage value be less than the interest rate.

This implies that under stumpage price uncertainty, optimal rotation length is longer than the deterministic optimal rotation length characterized by $G'(T) = rG(T)$ or $G'(T)/G(T) = r$, where the deterministic stumpage price/value is equal to the expected price $\bar{p}$/value $\bar{G}$. This result is supported by the finding of Norstrom [7] and may be due to the entrepreneur's desire for greater availability of inventory to meet uncertain future price.

Faustmann Many-Cycle Solution

Here, for the many-cycle Faustmann, the objective function to be maximized is equation (3) and the necessary condition for an optimum is

$$E\left\{ U'[V_1(T)]\,[G'(T) - \frac{1}{\lambda}G(T)] \right\} = 0. \tag{18}$$

where $\lambda = (1 - e^{-rT})/r$.

Using the same procedure as followed for the one-cycle case (with some additional terms), it can be shown that

$$E\left\{ U'[V_1(T)] \right\} [\bar{G}'(T) - \frac{1}{\lambda}\bar{G}(T)] < 0. \tag{19}$$

Given positive marginal utility, this implies that

$$\bar{G}'(T) < \frac{1}{\lambda}(\bar{G}(T)), \tag{20}$$

or

$$\bar{G}'(T)/\bar{G}(T) < \frac{1}{\lambda} r[1/(1 - e^{-rT})]. \tag{21}$$

Inequality equations (20) or (21) can then be called the Faustmann rotation rule under stumpage price uncertainty. Inequality equation (21) is identical with the inequality equation (17), excepting the term within the parentheses. Since the term within the parentheses is greater than one, the effective interest rate in equation (21) is greater than r. Thus, under conditions of certainty as well as under conditions of uncertainty, the Faustmann many-cycle rule implies a shorter rotation period than the Fisherian one-cycle rotation period. This occurs because the effective interest rate gets inflated in the former case. Inequality equation (21) also indicates that under stumpage price uncertainty, the optimal rotation length is longer than the deterministic rotation length characterized by

$$G'(T) = \frac{1}{\lambda}(G(T))$$

or

$$G'(T)/G(T) = \frac{1}{\lambda} = r(1 - 1/1 - e^{-rT}), \tag{22}$$

where the deterministic stumpage value is equal to the expected value G.

Effects of Increasing Risk

The effect (on the optimal rotation age) of an increasing risk in the sense of the effect of making a given probability distribution "slightly more risky" is considered in this section. For its simplicity and intuitive appeal, we consider the one-cycle Fisherian case only. It can easily be extended, with some more algebraic manipulations, to the Faustmann many-cycle case (where compared with the Fisherian result, the rotation age would be shorter). A small increase in risk is defined as a "stretching" of the probability distribution of the random variable, stumpage value (G(T)) around a constant mean equation [2].

To accomplish this, two slight parameters, one multiplicative (β) and one additive (θ), are used. Then, the stumpage value function can be expressed as $\beta G(T) + \theta$, and its discounted present value as $V_1(T) = [\beta G(T) + \theta] e^{-rT}$. θ is equivalent to an increase in the mean with all other moments constant. Because of the nonnegativity of G(T), an increase in β alone (from $\beta = 1$, $\beta = 0$) will increase the mean as well as the variance. To counteract this and preserve the mean (expected value), β is made to reduce simultaneously such that

$$dE[\beta G(T) + \theta] = E[G(T)d\beta + d\theta] = 0. \tag{23}$$

This implies that

$$\frac{d\theta}{d\beta} = -\bar{G}(T). \tag{24}$$

The one-cycle objective function

$$W_1(T) = E\left\{U[\beta G(T) + \theta)e^{-rT}]\right\}, \tag{25}$$

and the first-order condition of maximization with respect to T is

$$E\left\{U'(V_1)[\beta G'(T) - r(\beta G(T) + \theta)e^{-rT}]\right\} = 0. \tag{26}$$

Differentiating equation (26) implicity (when the solution T = T*) with respect to β, and evaluating the derivative at ($\beta = 1$, $\theta = 0$), and using equation (9) and equation (24) yields

$$\frac{\partial T^*}{\partial \beta} = \frac{e^{-rT}}{D}[E\left\{U''(V_1)(G(T) - \bar{G}(T)(G'(T) - rG(T))\right\}$$
$$+ E\left\{U'(V_1)(G'(T) - rG(T) + r\bar{G}(T))\right\}]. \tag{27}$$

So far, the forest manager's attitudes to risk has been restricted by the assumption of risk aversion only. To ascertain the sign of $\partial T^*/\partial\beta$, a further restriction on the utility function is imposed by means of the Arrow [4] and Pratt [5] "absolute risk aversion" function

$$R_A(V_1) = -\frac{U''(V_1)}{U'(V_1)}. \tag{28}$$

It is assumed here that the forest manager has a nonincreasing absolute risk aversion, i.e., $R_A(V_1)$ is a nonincreasing function of V_1. This assumption implies that as a decision maker becomes wealthier, the risk premium for any risky prospect, defined as the difference between the mathematical expectation of the return from the prospect and its certainty equivalent should decrease, or at least not increase.

In equation (27), since $e^{-rT}/D < 0$, the sign of $\partial T^*/\partial\beta$ is equivalent to the sign of the terms within the brackets. Consequently, the sign of this is investigated. Let us first consider the sign of the expression

$$E\left\{U''(V_1)(G'(T) - rG(T))\right\}, \tag{29}$$

drawn from equation (27). Let $\bar{V}_1(T)$ be the maximum level of discounted net returns, obtained when the rotation is optimal under conditions of certainty, i.e., when $G'(T) = rG(T)$. Then, since it is assumed that $R_A(V_1)$ is nonincreasing

$$R_A(V_1) \leqslant R_A(\bar{V}_1)$$

for

$$G'(T) - rG(T) \geqslant 0, \tag{30}$$

i.e., when the marginal gain is at least equal to the marginal loss.

Using equations (28) and (29), we have

$$-\frac{U''(V_1)}{U'(V_1)} \leqslant R_A(\bar{V}_1). \tag{31}$$

Since marginal utility is positive,

$$-U'(V_1)(G'(T) - rG(T)) \leqslant 0$$

for

$$G'(T) - rG(T) \geqslant 0. \tag{32}$$

Multiplying equation (31) by the left-hand side of equation (32), and taking the expected values, we obtain

$$E\left\{U''(V_1)(G'(T) - rG(T))\right\} \geqslant -R_A(\bar{V}_1)E\left\{U'(V_1)(G'(T) - rG(T))\right\} \tag{33}$$

since $R_A(V_1)$ is a given number. By the first-order condition equation (8), the right-hand side of equation (33) is equal to zero and, hence,

$$E\left\{U''(V_1)(G'(T) - rG(T))\right\} \geqslant 0. \tag{34}$$

Now, let us consider the sign of the whole expression of the first term within the brackets of equation (27) which may be written as

$$-\frac{1}{r}\left[E\left\{(G'(T) - rG(T))U''(V_1)(r\bar{G}(T) - rG(T))\right\}\right], \tag{35}$$

which, after some manipulations, yields

$$-\frac{1}{r}\left[E\left\{(G'(T) - rG(T))^2U''(V_1)\right\} - (\bar{G}'(T) - r\bar{G}(T)) \\ E\left\{(G'(T) - rG(T))U''(V_1)\right\} \\ - E\left\{(G'(T) - G'(T)\right\} E\left\{(G'(T) - rG(T))U''(V_1)\right\}\right]. \tag{36}$$

The first term of equation (36) within the brackets is positive, the second term is positive by equations (16) and (34), and the third term is zero by definition $(E\left\{(G'(T) - G'(T))\right\} = 0)$. Therefore, equation (36) is positive and, hence, the first term within the brackets of equation (27) is positive.

Let us now investigate the sign of the second term of equation (26) within the brackets, $E\left\{U'(V_1)(G'(T) - rG(T) + rG(T))\right\}$ which can be rewritten as

$$E\left\{U'(V_1)(G'(T) - rG(T)\right\} + rG(T)E\left\{U'(V_1)\right\}. \tag{37}$$

By the first-order condition equation (8), the first term of equation (37) is zero and by equation (4) the second term is positive. Therefore, equation (37) is positive and, hence, the second term within the brackets of equation (27) is also positive. Using these results and noting that $D < 0$ (from equation (9)), it follows that $\partial T^*/\partial\beta > 0$. Thus, nonincreasing absolute risk aversion is a sufficient condition for $\partial T^*/\partial\beta$ to be positive equation [7]. A positive $\partial T^*/\partial\beta$ implies that the impact of an increasing risk would be a lengthening of the optimal rotation age. Thus, the marginal impact of risk is expected to be identical (qualitatively) to the overall impact of the uncertain stumpage price.

CONCLUDING REMARKS

The present article is an attempt to apply the Sandmo model to answer a dominant question of forest economics: when to harvest a forest operated in an environment of stumpage price uncertainty. The results show qualitatively that the optimal rotation age of a forest would be longer under conditions of stumpage price uncertainty than under conditions of certainty. The model developed here abstracts from the nontimber multiple-use benefits (like recreation, flood control, and wildlife habitat) flowing from forest. Very often, unpredictable fluctuations in tree stock create uncertainty in such flows. Any realistic model of forest rotation will have to consider the net benefits of these flows as well as types of uncertainty associated with that.

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