J. ENVIRONMENTAL SYSTEMS, Vol. 20(2) 169-178, 1990-91

RISK ASSESSMENT WITH LOWER PROBABILITIES: APPLICATION TO TOXIC WASTES

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ABSTRACT

Traditional approaches to the quantitative analysis of uncertainty employ probabilities, although evidence to support the principled assessment of such additive measures is often lacking. We show how both optimization and conditionalization can be carried out using the less structured, hence more realistic, class of lower probabilities, illustrating these techniques with a problem involving toxic wastes.

Fragmentary data and incomplete evidence may often preclude a responsible analysis of risk in traditional probabilistic terms, that is, in terms of an *additive* measure of uncertainty. But adopting more realistic, less structured measures of uncertainty, while depriving us of some analytical tools of classical probability theory, need not leave us totally without analytical resources. In particular, both *optimization* and *conditionalization* can be successfully pursued when uncertainties are assessed in terms of lower probabilities with surprisingly weak structural

* Operated by Martin Marietta Energy Systems for the U.S. Department of Energy under Contract No. DE-AC05-84OR21400. The submitted manuscript has been authored by a contractor of the U.S. Government under contract No. DE-AC05-84OR21400. Accordingly, the U.S. Government retains a nonexclusive, royalty-free license to publish or reproduce the published form of the contribution, or allow others to do so, for U.S. Government purposes.

** Research supported in part by a grant from the University of Tennessee.

doi: 10.2190/KC2F-MA8A-DTRE-NQKR
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properties. These topics are treated, respectively, in the following two sections. The material on optimization is based on work of Dempster [1], Shapley [2], and Chateauneuf and Jaffray [3]. The material on conditionalization is based on work of Wagner [4], Wagner and Tonn [5], and Sundberg and Wagner [6], and represents a generalization of Jeffrey's [7, 8] theory of probability kinematics.

In this article we describe concrete applications of the aforementioned methods of optimization and conditionalization to some problems connected with a toxic waste dump. But the potential applications of these methods are, in our view, very extensive, including problems associated with power plant reliability, human exposure to carcinogens, global climatic change, and many types of policy analysis.

OPTIMIZATION WITH LOWER PROBABILITIES

Estimating the cost of cleaning up some body of toxic wastes is generally a problem of daunting complexity. The fragmentary nature of environmental data frustrates the more straightforward solutions to problems of this type. As we show in what follows, however, enough may be gleaned from such fragmentary data to enable us nevertheless to pursue a responsible analysis.

We wish to argue specifically here for the usefulness of certain set functions, called *lower probabilities*, in this enterprise. In order to highlight our main points, we shall make use of an extremely simple model of the problem in question. In particular, we assume that the total mass, M, of the body of wastes is known, as well as the set of types $T = \{t_1, \ldots, t_m\}$ of waste comprising this mass, along with the costs, $c(t_i)$, of cleaning up a unit mass of waste of type t_i. If the proportion of M attributable to waste of type t_i is known, and denoted by $q(t_i)$, and if a simple linear cost function is assumed, then the cost of cleaning up the entire body of waste will be

$$M \sum_{i=1}^{m} c(t_i)\, q(t_i) \tag{1}$$

We may of course be ignorant of the proportions $q(t_i)$. Suppose, however, that we are able to assess for each subset $A \subset T$ a *lower bound* $\ell(A)$ on the proportion of M attributable to wastes of all of the types belonging to A. Elementary consistency considerations dictate that any such set function should be subject to the restrictions

$$\ell(\emptyset) = 0, \tag{2}$$

$$\ell(T) = 1, \text{ and} \tag{3}$$

$$A_1 \cap A_2 = \emptyset \Rightarrow \ell(A_1 \cup A_2) \geq \ell(A_1) + \ell(A_2). \tag{4}$$

It is worth pointing out here that even in a state of total ignorance we may assess such an ℓ, simply setting $\ell(A) = 0$ for every proper subset of T and $\ell(T) = 1$, an unexceptionable (and uninformative) lower probability.

Of course, if we actually know the proportions $q(t_i)$, and extend q to subsets A of T by defining $q(A) = \sum_{t_i \varepsilon A} q(t_i)$, we shall be in possession of a particularly informative set function, namely, a probability measure. And this observation suggests a principled compromise with the approach of Eq. (1). Given a set function ℓ satisfying Eqs. (2), (3), and (4) above, identify all those probability measures q such that

$$q(A) \geq \ell(A) \qquad \text{for all } A \subset T, \tag{5}$$

and determine the minimum and maximum of the quantities given by Eq. (1), taken over all q satisfying Eq. (5), thereby bounding the clean-up costs.

The only problem is that if ℓ satisfies only Eqs. (2), (3), and (4), there may be *no* probability measure q satisfying Eq. (5). What additional properties of ℓ will ensure that the above approach may be activated? It follows from results of Shapley [2] and of Chateauneuf and Jaffray [3] that the additional restriction on ℓ,

$$\ell(A_1 \cup A_2) \geq \ell(A_1) + \ell(A_2) - \ell(A_1 \cap A2), \tag{6}$$

is sufficient to activate the above approach.

But what sorts of procedures ensure the assessment of an ℓ satisfying Eq. (6)? One way of arriving at such an ℓ (indeed, an ℓ with much stronger properties than Eq. (6)) is to make use of an old idea of Dempster [1], which has been somewhat obscured in its abstract formulation by Shafer [9]. Suppose that we have identified the set $S = \{s_1, \ldots, s_n\}$ of sources of the body of toxic wastes in question, and that we know the proportion $\pi(s_i)$ of M attributable to each source s_i. Suppose, in addition, that for each $s_i \varepsilon S$, we can identify a nonempty subset $A(s_i)$ of T, comprised of the possible types of waste attributable to source s_i. If we then define

$$\ell(A) = \sum_{\substack{s_i \varepsilon S: \\ A(s_i) \subset A}} \pi(s_i) \tag{7}$$

$\ell(A)$ will be a lower bound on the proportion of the total mass of the dump comprised of wastes in the class A. We remark that ℓ satisfies not just Eq. (6), but

$$\ell(A_1 \cup \ldots \cup A_r) \geq \sum_{\substack{I \subset \{1,\ldots,r\} \\ I \neq \emptyset}} (-1)^{|I|-1} \ (\bigcap_{i \varepsilon I} A_i) \quad \text{for all } r \geq 2. \tag{8}$$

In any case, the set of probability measures q satisfying Eq. (5) will be nonempty, and maximal and minimal values of the cost function Eq. (1), taken over all such q, may be easily calculated. It should be noted that while we have made no explicit mention of upper probabilities in the above discussion, they underlie our analysis in an implicit way. For if ℓ is a lower probability, its companion upper probability u is clearly defined by

$$u(A) = 1 - \ell(\overline{A}) \text{ for all } A \subset T. \tag{9}$$

For a pair of lower and upper probabilities ℓ and u, it is very easy to prove that a probability measure q satisfies Eq. (5) if and only if it satisfies

$$q(A) \leq u(A) \text{ for all } A \subset T, \tag{10}$$

making it superfluous to state the upper bound Eq. (10) as a constraint when optimizing Eq. (1).

Example 1. A toxic waste dump contains wastes of types $\{ t_1, t_2, t_3] = T$. The dump is known to have received a shipment s_1, accounting for 10 percent of its mass, a shipment s_2, accounting for 20 percent of its mass, and a shipment s_3, accounting for 50 percent of its mass. The set of sources $S = \{ s_1, s_2, s_3, s_4 \}$, with s_4 denoting the union of all remaining shipments, is naturally endowed with the probability $\pi(s_1) = 0.1$, $\pi(s_2) = 0.2$, $\pi(s_3) = 0.5$, and $\pi(s_4) = 0.2$. The specific composition of the individual shipments is unknown, but fragmentary records indicate that shipments s_1 and s_2 contained no wastes of type t_1, and shipment s_3 contained no wastes of type t_3. Here $A(s_1) = A(s_2) = \{ t_2, t_3 \}$, $A(s_3) = \{ t_1, t_2 \}$, and $A(s_4) = T$.

The lower probability ℓ defined by formula Eq. (7) takes the nonzero values $\ell(\{ t_2, t_3 \}) = 0.3$, $\ell(\{ t_1, t_2 \}) = 0.5$, and of course $\ell(T) = 1.0$. If the cost of cleaning up a unit mass of waste of type t_i is, let us say, i, for $i = 1,2,3$, then it can easily be seen that the cost function Eq. (1) is minimized over those q satisfying Eq. (5) when $q(t_1) = 0.7$, $q(t_2) = 0.3$, and $q(t_3) = 0$ and maximized when $q(t_1) = 0$, $q(t_2) = 0.5$, and $q(t_3) = 0.5$. This example is worked in more detail in Table 1.

Optimization is always nearly as simple as in the above example when ℓ satisfies Eq. (8) for all $r \geq 2$ (see [1]). When ℓ merely satisfies Eq. (6) the situation is somewhat more complex, but optima may still be found in a finite number of steps (see [2, 3]).

Table 1. Variables and Equations for Example 1

Types of Wastes:	t_1, t_2, t_3
Shipments of wastes to dump:	s_1, s_2, s_3, s_4
Total amount of waste in dump:	M (Assume = 1)
Proportion of waste (π) in dump by shipment:	$\pi(s_1) = 0.1, \pi(s_2) = 0.2, \pi(s_3) = 0.5, \pi(s_4) = 0.2$
Types of waste (A) in dump by shipment:	$A(s_1) = A(s_2) = \{t_2, t_3\}, A(s_3) = \{t_1, t_2\},$ $A(s_4) = \{t_1, t_2, t_3\}$
Costs of cleaning up (c) wastes by type:	$c(t_1) = 1, c(t_2) = 2, c(t_3) = 3$
Lower bound (ℓ) on proportion of total mass of dump attributable to wastes of a given set of types:	$\ell(t_1) = \ell(t_2) = \ell(t_3) = \ell(\{t_1\ t_3\}) = 0$ $\ell(\{t_2, t_3\}) = \pi(s_1) + \pi(s_2) = 0.3$ $\ell(\{t_1, t_2\}) = \pi(s_3) = 0.5$ $\ell(\{t_1, t_2, t_3\}) = \pi(s_1) + \pi(s_2) + \pi(s_3) + \pi(s_4) = 1.0$
Constraints on actual, but unknown, proportion (q) of total mass of dump attributable to wastes of a given type:	$q(t_i) \geq 0, i = 1,2,3$ $q(t_1) + q(t_2) + q(t_3) = 1$ $q(t_1) + q(t_2) \geq 0.5$ $uq(t_2) + q(t_3) \geq 0.3$
Optimization Problems:	MINIMIZE (respectively, MAXIMIZE) $C = q(t_1) + 2q(t_2) + 3q(t_3)$ SUBJECT TO ABOVE CONSTRAINTS on q
Solutions:	$c_{MIN} = (0.7) + 2 * (0.3) + 3 * (0.0) = 1.3$ $c_{MAX} = (0.0) + 2 * (0.5) + 3 * (0.5) = 2.5$

CONDITIONALIZATION WITH LOWER PROBABILITIES

Let p be a probability measure defined on subsets of the set T. If additional evidence indicates that the true state of affairs lies in the subset E of T, it is often[1] appropriate to update p to the probability measure q, where $q(A) = p(A|E) := p(A \cap E)/p(E)$ for all $A \subset T$.

A generalization of this updating method, due to Jeffrey [7], starts with additional evidence of a less dogmatic sort. In Jeffrey's generalization of simple conditionalization there is a pairwise disjoint collection $\mathcal{E}$ of subsets of T and a collection $\{ \mu_E : E \in \mathcal{E} \}$ of positive real numbers summing to one. New evidence establishes that possible revisions of p be restricted to those q satisfying

$$q(E) = \mu_E \text{ for all } E \in \mathcal{E}. \tag{11}$$

If it is judged in addition that any acceptable revision q of p should satisfy

$$q(A|E) = p(A|E) \text{ for all } A \subset T \text{ and all } E \in \mathcal{E}, \tag{12}$$

then there is a uniquely acceptable revision q of p, defined for all $A \subset T$ by

$$q(A) = \sum_{E \in \mathcal{E}} \mu_E \, p(A|E). \tag{13}$$

Evidence may of course fail to establish the sorts of restrictions on q embodied in Eq. (11), their substantial weakening of the dogmatic condition $q(E) = 1$ notwithstanding. In seeking to equip certain expert systems with uncertainty management capabilities, we have become convinced of the need for an approach to updating in which the possible revisions q of a prior p are subject to the restriction

$$q(A) \geq \ell(A) \text{ for all } A \subset T, \tag{14}$$

where ℓ is a Dempsterian lower probability arising as in Eq. (7). In the analysis of this problem it has turned out to be fruitful to make use of the "Möbius transform" m of ℓ, defined by the slight variant of Eq. (7),

$$m(A) = \sum_{\substack{s_i \in S: \\ A(s_i) = A}} u(s_i) \tag{15}$$

or, equivalently, by

$$m(A) = \sum_{E \subset A} (-1)^{|A-E|} \ell(E). \tag{16}$$

[1] That one cannot *mechanically* condition on E upon finding out that the truth lies in E is, fortunately, gaining wider recognition. The notorious three prisoners problems (see Diaconis and Zabell [10] or Jeffrey [8] for delightful discussions of this problem, originally due to Martin Gardner) illustrates the pitfalls of mechanical conditionalization.

The mapping m is termed a *basic probability assignment* in the work of Shafer [9]. It is in any case easy to show that $m : 2^T \rightarrow [0, 1]$, with $m(\emptyset) = 0$, and that

$$\sum_{E \subset A} m(E) = \ell(A) \quad \text{for all } A \subset T. \tag{17}$$

In particular,

$$\sum_{E \subset T} m(E) = \ell(T) = 1. \tag{18}$$

A subset $E \subset T$ for which $m(E) \neq 0$ is called an *evidentiary focal element* and $\mathcal{E}$ now represents the collection of evidentiary focal elements. Just as Eq. (11) and Eq. (12) yield Eq. (13), Eq. (14), combined with a condition generalizing Eq. (12) (see [4]), now yields the updating formula

$$q(A) = \sum_{E \in \mathcal{E}} m(E)\, p(A|E). \tag{19}$$

Eq. (19) reduces to Jeffrey's Eq. (13) when evidentiary focal elements are pairwise disjoint.

We remark that Eq. (19) may also be viewed as a way of *upgrading* the lower probability ℓ to a probability measure q by drawing on a probability p, which in practice often records past relative frequencies. This point of view is explored in detail in Wagner and Tonn [5]. We note also that Sundberg and Wagner [6] have proved that formula Eq. (19) actually produces a probability measure q bounded below by ℓ so long as ℓ merely satisfies Eq. (6) and m is defined by Eq. (16). We conclude with an application of Eq. (19) to an extension of Example 1.

Example 2. Suppose that the dump of Example 1 accepted wastes from an identifiable set of chemical factories. In combination, the factories are known to have produced wastes of type t_1, t_2, and t_3, with $p(t_1) = 0.4$, $p(t_2) = 0.3$, and $p(t_3) = 0.3$ being the proportions of the total mass of wastes produced by these factories attributable to the three types. With no further evidence regarding the composition of the dump in question, we might employ p as an estimate of the proportions of the types of waste in the dump.

Suppose, however, that we are apprised of the information about shipments to this dump, as given in Example 1. As indicated in that example, this information puts the restrictions $q(\{t_1, t_2\}) \geq \ell(\{t_1, t_2\}) = 0.5$, and $q(\{t_2, t_3\}) \geq \ell(\{t_2, t_3\}) = 0.3$ on any probability q representing the proportions of the various types of wastes in the dump. To activate formula Eq. (19) we must judge that within the aggregate of

Table 2. Variables and Equations for Example 2

Types of wastes in dump:	t_1, t_2, t_3
Shipments of wastes to dump:	s_1, s_2, s_3, s_4
Total amount of waste in dump:	M (Assume = 1)
Proportion of waste (π) in dump by shipment:	$\pi(s_1) = 0.1, \pi(s_2) = 0.2, \pi(s_3) = 0.5, \pi(s_4) = 0.2$
Types of waste (A) in dump by shipment:	$A(s_1) = A(s_2) = \{t_2, t_3\}, A(s_3) = \{t_1, t_2\},$ $A(s_4) = \{t_1, t_2, t_3\}.$
Proportions of waste (p) produced by the factories by type:	$p(t_1) = 0.4, p(t_2) = 0.3, p(t_3) = 0.3$
Basic probability assignments (m) of sets of types of wastes:	$m(t_1) = m(t_2) = m(t_3) = m(\{t_1, t_3\}) = 0$ $m(\{t_2, t_3\}) = \pi(s_1) + \pi(s_2) = 0.3$ $m(\{t_1, t_2\}) = \pi(s_3) = 0.5$ $m(\{t_1, t_2, t_3\}) = \pi(s_4) = 0.2$

Revision of p to updated estimate of proportions of types of wastes (q) using shipment information (as expressed in m) and the conditionalization methods of Wagner and Tonn [5]:

$$q(t_1) = m(\ t_1,t_2\ ) * \frac{p(t_1)}{p(t_1)+p(t_2)} + m(\ t_1,t_2,t_3\ ) * \frac{p(t_1)}{p(t_1)+p(t_2)+p(t_3)} =$$

$$0.5 * \frac{0.4}{0.4+0.3} + 0.2 * \frac{0.4}{0.4+0.3+0.3} = .37$$

$$q(t_2) = m(\ t_2,t_3\ ) * \frac{p(t_2)}{p(t_2)+(t_3)} + m(\ t_1,t_2\ ) * \frac{p(t_2)}{p(t_1)+p(t_2)} +$$

$$m(\ t_1,t_2,t_3\ ) * \frac{p(t_2)}{p(t_1)+p(t_2)+p(t_3)} = 0.3 * \frac{0.3}{0.3+0.3} + 0.5 * \frac{0.3}{0.4+0.3} +$$

$$0.2 * \frac{0.3}{0.4+0.3+0.3} = .42$$

$$q(t_3) = m(\ t_2,t_3\ ) * \frac{p(t_3)}{p(t_2)+p(t_3)} + m(\ t_1,t_2,t_3\ ) * \frac{p(t_3)}{p(t_1)+p(t_2)+p(t_3)} =$$

$$0.3 * \frac{0.3}{0.3+0.3} + 0.2 * \frac{0.3}{0.4+0.3+0.3} = .21$$

shipments s_1 and s_2, wastes of type t_2 and t_3 may reasonably be assumed to be represented in proportion to the quantities $p(t_2)$ and $p(t_3)$,[2] and, similarly, that within shipment s_3 wastes of type t_1 and t_2 are represented in proportion to $p(t_1)$ and $p(t_2)$, and within "shipment" s_4 wastes of type t_i are represented in proportion to $p(t_i)$ for $i = 1,2,3$. So judging, we are warranted in employing Eq. (19) to construct a probability measure q on T. Using $m(\{t_1, t_2\}) = 0.5$, $m(\{t_2, t_3\}) = 0.3$, and $m(T) = 0.2$, we have, for example, $q(t_3) = (0.5)(0) + (0.3)(0.5) + (0.2)(0.3) = 0.21$. Complete details for this example appear in Table 2. Note that q can be viewed either as an updating of the "prior" probability p or as an upgrading of the lower probability ℓ.

DISCUSSION

Many lines of inquiry need to be pursued to make optimal use of the methodology described in this article. The issues requiring further investigation are both practical (methods for eliciting lower probabilities and communicating them to the public and to decision makers) and theoretical (methods for combining lower probabilities derived from various pieces of evidence and the articulation of rigorous criteria for choosing among such methods; methods of updating or upgrading a more general class of lower probabilities than those treated in this article). The generalizations of probability theory required to represent what we know of chance and risk in an honest and responsible way will, we think, go far beyond earlier generalizations (such as Dempster-Shafer Theory, concerned exclusively with lower probabilities ℓ satisfying Eq. (8) for *all* $r \geq 2$), the pioneering style and verve of those early generalizations notwithstanding.

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[2] This condition will of course be met if in each of the separate shipments s_1 and s_2 wastes of type t_2 and t_3 are represented in proportion to the quantities $p(t_2)$ and $p(t_3)$, but an assumption of this strength is not necessary.

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