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TRANSIENT RESPONSE OF LAKE SYSTEMS TO PHOSPHORUS*

BABAK NAGHAVI
Louisiana Transportation Research Center

DONALD D. ADRIAN
Louisiana State University, Baton Rouge

ABSTRACT

Impoundments and lake systems become eutrophic through excessive nutrient (phosphorus and nitrogen) loadings. Deterioration of water quality because of algal blooms has forced many water quality control boards to adopt measures to reduce discharge of such nutrients into the water bodies. Phosphorus is the nutrient that is selected most frequently for control as there are fewer pathways for it to enter a lake than there are for nitrogen. For instance, the Virginia Water Control Board has adopted an average monthly limit of 2 mg/l of total phosphorus in the effluent of about forty wastewater treatment plants in the Chesapeake Bay Drainage Basin [1].

A practical model to compute the transient response of a lake system to phosphorus is of great benefit to water resources planners and designers. Many systems of interest in environmental engineering are described by first order linear differential equations which have readily obtainable solutions when the coefficients are constant. Simultaneous differential equations occur less frequently than single differential equations and the methods for solving systems of these equations are less commonly known to engineers. Continuous, lumped parameter, linear systems of n ordinary differential equations are written

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$$\frac{dx_1(t)}{dt} = f_{11}(t)x_1(t)+\ldots+f_{1n}(t)x_n(t) + g_{11}(t)u_1(t)+\ldots+g_{1m}(t)u_m(t)$$

$$\vdots$$

$$\frac{dx_n(t)}{dt} = f_{n1}(t)x_1(t)+\ldots+f_{nn}(t)x_n(t) + g_{n1}(t)u_1(t)+\ldots+g_{nm}(t)u_m(t) \quad (1)$$

where n and m are the dimensions of the state and input vectors

$$\mathbf{x}(t) = [x_1(t),\ldots,x_n(t)]^T \quad (2)$$

and

$$\mathbf{u}(t) = [u_1(t),\ldots,u_n(t)]^T \quad (3)$$

These equations can be written in the compact form

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) \quad (4)$$

where $\dot{\mathbf{x}}(t)$ stands for $d\mathbf{x}(t)/dt$ and $\mathbf{F}(t)$ and $\mathbf{G}(t)$ are the two matrices $[f_{ij}(t)]$ and $[g_{ij}(t)]$ of dimension m x n and n x m, respectively.

Certain subclasses of the general problem described above are of particular interest to environmental engineers. Of special interest are equations having constant coefficients in which $[f_{ij}(t)]$ consists of constants $[f_{ij}]$ and $[g_{ij}(t)]$ consists of constants $[g_{ij}]$. In addition, homogeneous equations occur when $u_l(t) = \ldots = u_m(t) = 0$. In many applications, we are interested in the solution given when $\mathbf{G}(t)$ is the identity matrix. A theorem which is applicable to both a single linear ordinary differential equation and to a system of linear ordinary differential equations is that the most general solution of the nonhomogeneous Equation (4) is obtained by adding to any particular solution of Equation (4) the general solution of the homogeneous equation [2].

PURPOSE AND SCOPE

The purpose of the article is to discuss alternative methods for solving systems of equations which arise when describing a transient lake eutrophication problem. We restrict our discussion to a comparison of methods for solving systems of ordinary linear differential equations having constant coefficients. We illustrate application of the methods with a system of three simultaneous equations which describe lake phosphorus dynamics. We will examine five methods for solving systems of equations: reducing the number of equations to a single higher order equation, solving by the eigenvalue method, using Laplace transforms, developing a trial function, and obtaining the solution by series. We will not discuss numerical methods as this subject is covered by extensive literature elsewhere [3, 4]. Also,

we do not discuss the method of eliminating the independent variable by combining equations as it eliminates the time variable [5].

PHOSPHORUS MODEL FOR LAKE EUTROPHICATION

Dynamic models for prediction of equilibrium phosphorus concentrations in a lake system require the solution of three simultaneous, linear, first order differential equations which are derived from the lake model shown in Figure 1. The lake model as presented by Bingham and Feng [6], Havis [7], and Havis and Ostendorf [8] is

Sediment solid phase phosphorus

$$\frac{dPs}{dt} = -k_3 P_s + \frac{D_L}{D_r} k_2 P_L \tag{5}$$

Sediment interstitial phosphorus

$$\frac{dP_i}{dt} = \frac{k_3}{\varepsilon} P_s - \frac{k_1}{D_r}(P_i - P_L) \tag{6}$$

Lake water phosphorus

$$\frac{dP_L}{dt} = \frac{P_o}{T} - \frac{P_L}{T} + \frac{\varepsilon k_1}{D_L}(P_i - P_L) - k_2 P_L \tag{7}$$

where

P_s = sediment solid phase phosphorus concentration, μg/l
P_i = sediment interstitial water phosphorus concentration, μg/l
P_L = lake water phosphorus concentration, μg/l
k_1 = film diffusion coefficient, m/day
k_2 = sedimentation rate coefficient, day^{-1}
k_3 = desorption rate coefficient, day^{-1}
D_L = average lake depth, m
D_r = sediment reactive depth, m
ε = sediment porosity
T = V_L/Q = lake detention time, days
V_L = lake volume, m^3
Q = flow rate into lake, m^3/day
P_o = average annual influent total phosphorus concentration, μg/l

Removal of the nonhomogeneous part of the lake model problem is accomplished by introducing new dependent variables $h_s(t)$, $h_i(t)$, and $h_L(t)$ which describe the departure from steady state conditions.

$$h_s(t) = P_s(t) - P_{s\infty} \tag{8}$$

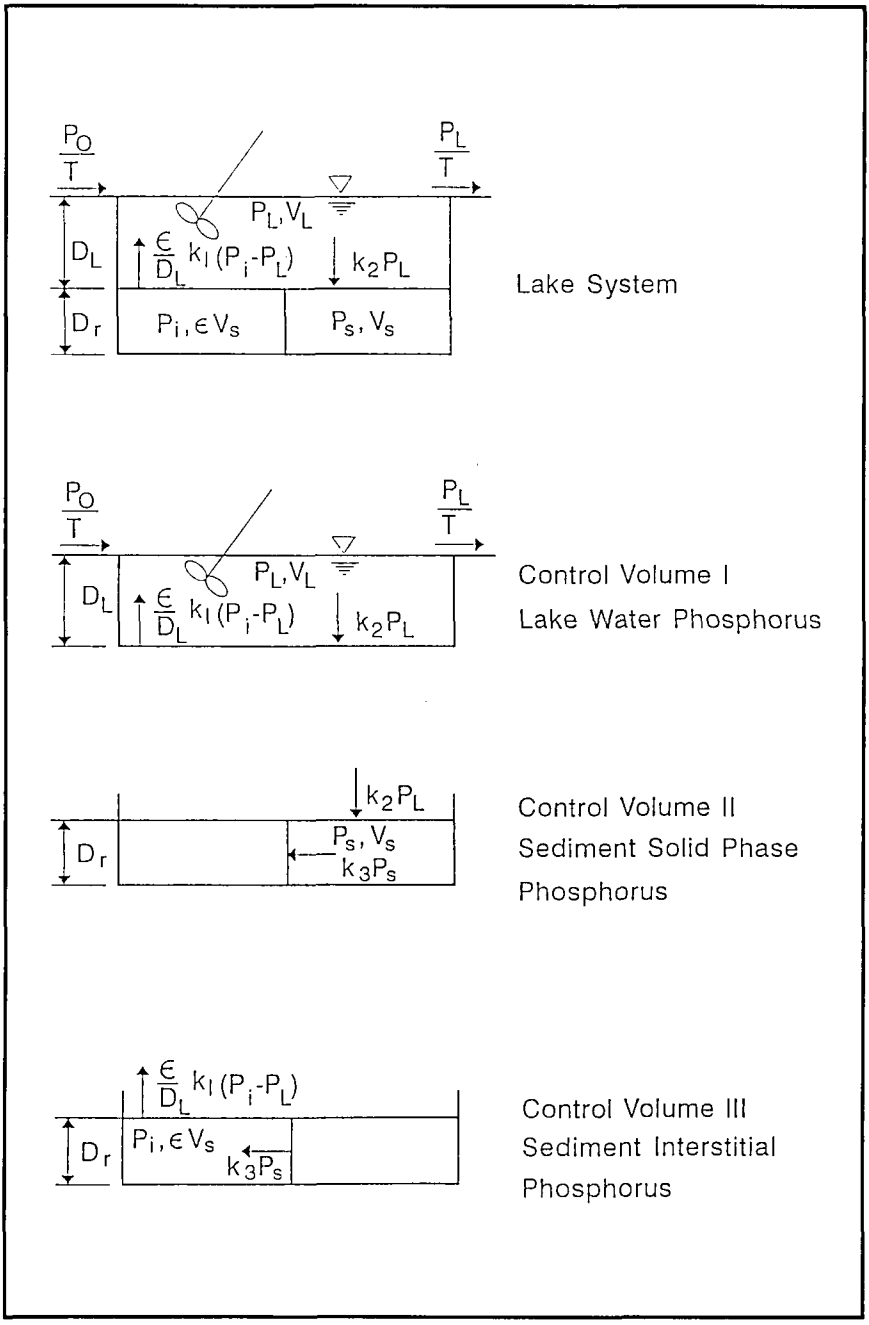

Figure 1. Control volumes used in dynamic lake phosphorus model.

$$h_i(t) = P_i(t) - P_{i\infty} \tag{9}$$

$$h_L(t) = P_L(t) - P_{L\infty} \tag{10}$$

where

$h_s(t)$ = sediment solid phase phosphorus concentration variable, μg/l
$h_i(t)$ = interstitial phosphorus concentration variable, μg/l
$h_L(t)$ = lake water phosphorus concentration, μg/l
$P_{s\infty}$ = sediment solid phase concentration as time approaches infinity, μg/l
$P_{i\infty}$ = sediment interstitial phosphorus concentration as time approaches infinity, μg/l
$P_{L\infty}$ = lake water phosphorus concentration as time approaches infinity, μg/l

The steady state conditions are found by setting the left-hand side of Equations (5), (6), and (7) equal to zero, solving them simultaneously and obtaining

$$P_{s\infty} = \left[\frac{k_2 D_L}{k_3 D_r}\right] P_o \tag{11}$$

$$P_{i\infty} = \left[\frac{k_2 D_L}{k_1 \varepsilon}\right] P_o + P_o \tag{12}$$

$$P_{L\infty} = P_o \tag{13}$$

The homogeneous system of differential equations expressed in matrix notation is

$$\begin{bmatrix} \frac{dh_s}{dt} \\ \frac{dh_i}{dt} \\ \frac{dh_L}{dt} \end{bmatrix} = \begin{bmatrix} -k_3 & 0 & \frac{k_2 D_L}{D_r} \\ \frac{k_3}{\varepsilon} & -\frac{k_1}{D_r} & \frac{k_1}{D_r} \\ 0 & \frac{\varepsilon k_1}{D_1} & -\left(\frac{1}{T} + \frac{\varepsilon k_1}{D_L} + k_2\right) \end{bmatrix} \begin{bmatrix} h_s \\ h_i \\ h_L \end{bmatrix} = \begin{bmatrix} a_1 & 0 & a_2 \\ a_3 & a_4 & a_5 \\ 0 & a_6 & a_7 \end{bmatrix} \begin{bmatrix} h_s \\ h_i \\ h_L \end{bmatrix} \tag{14}$$

where $a_1,...,a_7$, equal the corresponding terms in the coefficient matrix. The vector form of Equation (14) is

$$\dot{\mathbf{h}}(t) = \mathbf{A}\mathbf{h}(t) \tag{15}$$

SOLVING LAKE SYSTEM EQUATIONS

Five methods of solving systems of differential equations will be discussed in turn and will be illustrated by application to the transient lake phosphorus model.

The methods will be discussed in the order: 1) reducing the number of equations to a single higher order equation, 2) solving by the eigenvalue method, 3) using Laplace transforms, 4) developing a trial function, and 5) obtaining the series solution.

Reduction to a Single Higher Order Equation

This method is a relatively simple approach to systems of differential equations and is widely referenced. Unknown functions and their derivatives are successfully eliminated until a single higher order differential equation containing only one unknown function and its derivatives is obtained. This equation is solved, and then other unknown functions are found [9-12].

Eigenvalue Method

In this method, vectors and matrices are used to solve problems of linear differential equations [10, 13, 14]. A system of differential equations can be written

$$\dot{\mathbf{h}} = \mathbf{A}\mathbf{h} \tag{16}$$

where $\mathbf{A}$ is the coefficient matrix and $\dot{\mathbf{h}}$, ($d\mathbf{h}/dt$), and $\mathbf{h}$ are vectors. By postulating the solution $\mathbf{h} = \mathbf{Z}\, e^{\lambda t}$, and canceling out $e^{\lambda t}$, we obtain

$$\mathbf{AZ} = \lambda\mathbf{Z} \; or \; (\mathbf{A} - \lambda\mathbf{I})\mathbf{Z} = 0 \tag{17}$$

The characteristic equation when $\mathbf{Z} \neq 0$ is

$$det\,(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{18}$$

The roots of the characteristic equation are then used in Equation (17) to solve for $\mathbf{Z}$. A value of λ for which Equation (17) has a solution $\mathbf{Z} \neq 0$ is called an eigenvalue of matrix $\mathbf{A}$, and $\mathbf{Z}$ is called an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ. A linear combination of the independent solutions is also a solution.

Laplace Transform Method

The Laplace transformation can be used for solving systems of differential equations [15, 16, 17]. For f(t), a given function which is defined for all $t \geq 0$, the Laplace transform is given as

$$F(s) = \int_0^\infty e^{-st} f(t)dt \tag{19}$$

When solving a system of differential equations, after applying the transform, a solution in terms of s can be found for the transform F(s) by elimination. Then by taking the inverse transform of F(s), the original function f(t) can be obtained.

Tables of functions f(t) and their Laplace transforms F(s) can be found in many mathematics textbooks and reference books [18, 19, 20].

Trial Function Method

The trial function method [9] is a non-elegant method which requires little background on the part of the user. It is closely related to the method of undetermined coefficients. Havis [10] applied it to solve the problem defined by Equations (5), (6), and (7). One assumes a solution such as $\mathbf{h} = \mathbf{z}e^{\lambda t}$ as done in the eigenvalue method, but finds the coefficients and exponents that are appropriate for the problem being solved by substituting the assumed solution into the differential equations.

Series Method

This method is presented in Hurewicz [2] and is explained in detail in Frazer et al. [21] and in Rinaldi et al. [22]. One advantage that this method has over the others is its capability of handling large systems of differential equations. However, one may experience instability or slow convergence when the independent variable (time) becomes large. Hurewicz hints about the instability of the method as the series converges uniformly in a sufficiently small neighborhood of the initial point [2]. The instability problem may be resolved by keeping the number of terms in the series small by re-initializing the solution periodically. The solution to a system of homogeneous differential equations with initial time equal to zero can be given as

$$\mathbf{h}(t) = e^{\mathbf{A}t}\mathbf{h}(0) \tag{20}$$

where

$$e^{\mathbf{A}t} = I + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^3}{3!} + \ldots \tag{21}$$

In Equation (21), **I** is the identity matrix and **A** is the nxn matrix of constant coefficients, such as the coefficient matrix in Equation (14). Equation (21) can become cumbersome for large values of time as more terms are needed to reach a desired accuracy.

RESULTS

The solution to the system of differential equations (Equation 14), and their initial conditions are shown in detail in Appendix I. When the parameters in Appendix I are applied, the four methods of solution previously discussed yield

$$P_s(t) = 119{,}025.90\, e_{\lambda 1t} - 168.44 e_{\lambda 2t} - 1.16 e_{\lambda 3t} + 148{,}833.60 \tag{22}$$

$$P_i(t) = 179.77e_{\lambda 1t} + 21.33e_{\lambda 2t} - 5.30e_{\lambda 3t} + 244.70 \quad (23)$$

$$P_L(t) = 24.39e_{\lambda 1t} + 15.23e_{\lambda 2t} + 0.38e_{\lambda 3t} + 50.00 \quad (24)$$

The solution by the series method can be written as

$$\begin{bmatrix} h_s(t) \\ h_i(t) \\ h_L(t) \end{bmatrix} = e^{At} \begin{bmatrix} h_{so} \\ h_{io} \\ h_{Lo} \end{bmatrix} \quad (25)$$

where e^{At} can be obtained from Equation (21) and **A** (the coefficient matrix) is

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & a_2 \\ a_3 & a_4 & a_5 \\ 0 & a_6 & a_7 \end{bmatrix} \text{ and } \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A solution for the series (Equation 21) can be easily obtained for a given accuracy using a computer program to solve for $h_s(t)$, $h_i(t)$, and $h_L(t)$. This solution is accurate when the time t is small enough that the series converges rapidly. However, for a large number of terms, as would be encountered when projecting the solution far into the future where t is large, the solution becomes unstable. Keeping the number of terms small by re-initializing the solution periodically will avoid the instability problem. We discuss this issue in detail when discussing the results.

Numerical results for a lake phosphorus dynamics problem are calculated in Appendix III and the results are shown in Table 1. All of the methods produced

Table 1. Lake Phosphorus Dynamics Results

Time (days) (1)	P_L (μg/l) (2)	P_i (μg/l) (3)	P_s (mg/g-dry) (4)	No. of Terms (5)
0	90.00	440.50	2.68	2
1	86.15	438.85	2.68	6
3	81.17	433.73	2.68	12
5	78.31	429.90	2.68	17
10	75.35	425.50	2.67	30
100	73.49	417.90	2.63	62[a]
1000	66.55	366.69	2.29	62[a]
5000	53.47	270.30	1.66	62[a]
10000	50.47	248.18	1.51	62[a]
∞	50.00	244.70	1.49	N/A

[a]Values were obtained by re-initialization at twenty-five day periods as the series solution became unstable otherwise.

identical results. Again, however, the series method without re-initialization became unstable as time became larger. Re-initialization of the series at twenty-five-day intervals when the series required sixty-two terms to converge allowed the results to be calculated to 10,000 days, producing results identical to those obtained by other methods.

None of the textbooks which were reviewed contained all of the methods which we evaluated. Most books discussed only one or two of the methods. The older books presented the method of reduction to a single differential equation. Books written in the 1950s introduced the eigenvalue method, sometimes to the exclusion of other methods. The Laplace transform method was presented in operational mathematics books. Explicit references for the trial function method are few, although one can gain insight into the method by studying the method of undetermined coefficients. The series method, associated with matrix reference books for many years, is now more widely referenced. Its popularity is associated with the availability of computers for carrying out the tedious calculations.

All of the methods except the series method involve solutions for the roots of a polynomial equation having the same degree as there are simultaneous differential equations. Solution for the roots of the polynomial is made less difficult on a computer by the use of numerical algorithms. The question of which method is easiest arises. We rank the methods in the order of increasing difficulty as:

1. series method (if one is alert to convergence and instability problems),
2. reduction to a single higher order equation,
3. the eigenvalue method,
4. the trial function method, and
5. the Laplace transform method.

However, the ranking is subjective and may change depending upon one's experience. For example, one of the difficulties with the trial function method is that one has to prescribe the form of solution. If a term is not prescribed when one is selecting a trial function, trouble will be encountered later in the solution procedure. Laplace transforms become easier to use as one gains experience with them. The series method is attractive as one can write the form of the solution, Equation (20), as soon as the differential equation, Equation (15), and the initial condition are developed. All of the methods produced the same results. Care had to be exercised, however, with the series method as it showed signs of instability as the number of terms in the series increased. Re-initialization of the series avoided the instability problem, so it produced accurate results.

Doetsch cautions that when more than three equations are involved, the method of combining the equations into a single higher order equation (called the classical method) becomes impractical and the Laplace transform method is superior [23]. In fact, he states, "[The Laplace transform method] shows its full power in the solution of several differential equations where it leads to much greater insight and much less calculation than the classical method, which is not practical at all"

[23, p. 76]. He points out that the Laplace transform method can handle functions which are not differentiable, whereas the classical method of reduction to a single equation will fail. He also notes the initial conditions are easier to introduce in the Laplace transform method.

CONCLUSIONS

Table 1 shows that the phosphorus concentration in the lake water adjusted rapidly to the change in input phosphorus concentration. During the first year the lake phosphorus concentration made about half its ultimate adjustment. The sediment serves as a reservoir for phosphorus. It slowly adjusts to the change in input conditions, and feeds phosphorus through the interstitial water to the lake water for many years.

Experience and personal preference are the major reasons one would select one method of solving the equations describing phosphorus dynamics in a lake in place of another. The series method is the recommended method because it is easier to set up. However, one has to be alert to avoid slow convergence of the series and instability if convergence requires too many terms.

LIST OF SYMBOLS

A	= coefficient in equation, lake surface area;
$\mathbf{A}$	= coefficient matrix;
D_L	= average depth of lake;
D_r	= sediment reactive depth;
$\mathbf{F}(t)$	= coefficient matrix;
$F(s)$	= Laplace transform;
$f_{ij}(t)$, $g_{ij}(t)$, $h_{ij}(t)$	= coefficients in matrix;
$f(t)$	= function of time in Laplace transforms;
$\mathbf{G}(t)$	= coefficient matrix;
$h_i(t)$, $h_L(t)$, $h_s(t)$	= variable for interstitial, lake water, and sediment phosphorus
$\mathbf{I}$	= identity matrix;
k_1	= film diffusion coefficient;
k_2	= sedimentation rate coefficient;
k_3	= desorption rate coefficient;
P_i, P_L, P_o, P_s	= total phosphorus concentration in sediment interstitial water, lake water, influent stream, and sediment solid phase;
$P_{i\infty}$, $P_{L\infty}$, $P_{s\infty}$	= steady state sediment interstitial, lake water, and solid phase total phosphorus concentration;
Q	= flow rate into lake;
s	= parameter in Laplace transform;
T	= transpose of matrix, lake detention time;
t	= time;

$u_j(t)$ = function in nonhomogeneous equation;
V_L, V_s = volume of lake and volume of solids;
$\mathbf{x}(t)$ = matrix of functions;
x = variable in differential equation;
$x_1(t),...,x_n(t)$ = functions in matrix;
$a_1,...,a_7$ = coefficients in matrix;
$\mathbf{Z}$ = eigenvector of the coefficient matrix;
$\lambda_1, \lambda_2, \lambda_3$ = roots of polynomial equation;
ε = porosity;
e^{Ft} = series expansion of the matrix $\mathbf{F}$ for time;
det( ) = determinant

APPENDIX 1—PHOSPHORUS DYNAMICS PROBLEM

The following numerical values are taken from Havis [7] for Lake Warner, Hadley, MA. P_{Lo} = 90.0 μg/l, P_{io} = 440.5 μg/l, P_{so} = 2.6769 mg/g-dry (267,690 μg/l), P_o = 50.0 μg/l, k_1 = 0.091 m/day, k_2 = 0.176 day^{-1}, k_3 = 0.001 day^{-1}, V_L = 4.35×10^5 m^3, D_L = 1.691 m, D_r = 0.1 m, Q = 4.89×10^4 m^3/day, ε = 0.84, $a_1 = -k_3$ = 0.0010 day^{-1}, $a_2 = (k_2D_L)/(D_r)$ = 2.9767 day^{-1}, $a_3 = k_3/\varepsilon$ = 0.0012 day^{-1}, $a_4 = -k_1/D_r$ = -0.91 day^{-1}, $a_5 = k_1/D_r$ = 0.91 day^{-1}, $a_6 = (\varepsilon k_1)/D_L$ = 0.0452 day^{-1}, and $a_7 = -[1/T + (\varepsilon k_1)/D_L + k_2]$ = -0.3336 day^{-1}.

Also, we have

$$h_s(t) = P_s(t) - P_{s\infty} = P_s(t) - \left(\frac{k_2 D_L}{k_3 D_r}\right) P_o = P_s(t) - 148{,}833.60\ \mu g/l \tag{26}$$

$$h_i(t) = P_i(t) - P_{i\infty} = P_i(t) - \left(\frac{k_2 D_L}{k_1 \varepsilon}\right) P_o - P_o = P_i(t) - 244.70\ \mu g/l \tag{27}$$

$$h_L(t) = P_L(t) - P_{L\infty} = P_L(t) - P_o = P_L(t) - 50.00\ \mu g/l \tag{28}$$

Therefore, we need to solve

$$\begin{bmatrix} \frac{dh_s}{dt} \\ \frac{dh_i}{dt} \\ \frac{dh_L}{dt} \end{bmatrix} = \begin{bmatrix} -0.001 & 0 & 2.9767 \\ 0.0012 & -0.91 & 0.91 \\ 0 & 0.0452 & -0.3336 \end{bmatrix} \begin{bmatrix} h_s \\ h_i \\ h_L \end{bmatrix} \tag{29}$$

subject to the initial conditions, h_{so} = 118.856.60 μg/l, h_{io} = 195.80 μg/l, and h_{Lo} = 40.00 μg/l.

Solution by the Series Method

Solution by the series method is reached by the use of Equation (20) where e^{At} is obtained from Equation (21) and **A** is the coefficient matrix. Results of the computer solution are identical to the analytical solutions obtained from other methods and are shown in Table 1.

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Direct reprint requests to:

Dr. Babak Naghavi
Louisiana Transportation Research Center
4101 Gourrier Avenue
Baton Rouge, LA 70808