J. ENVIRONMENTAL SYSTEMS, Vol. 27(2) 113-125, 1999

ENVIRONMENTAL PROJECT EVALUATION BASED ON FUZZY RELATIONAL EQUATIONS

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ABSTRACT

The article examines the use of fuzzy relational equations in the context of the evaluation of projects with environmental consequences such as road and transit projects. Projects are characterized in terms of linguistic expressions of "performance" with respect to impacts or factors and the "importance" of those impacts or factors. An illustrative example is developed.

INTRODUCTION

Recently fuzzy methods of environmental project evaluation have been proposed which more adequately acknowledge the uncertainty (vagueness) and imprecision characteristic of projects (for example [1, 2]). Uncertainty may be represented in terms of *fuzzy sets.* Smith [2] illustrates *fuzzy additive weighting* in which the outcomes of projects with respect to a given criterion or factor (e.g., "savings in travel time," "social dislocation," "wildlife impact") may be represented in terms of (continuous) fuzzy subsets (*fuzzy numbers*) representing linguistic values (e.g., "low," "medium," "high") of a linguistic variable "performance." In addition, weights for criteria (factors) may be expressed in terms of a linguistic variable "importance" again assuming values (e.g., "low," "medium," "high") represented by fuzzy numbers.

Smith [3] illustrates a *fuzzy rule-based system* for the environmental evaluation of projects which are assessed relative to each other with respect to criteria (factors). This system consists of rules expressed as conditional propositions of the general form "if V is **A** the U is **B**" where V and U are linguistic variables and **A** and **B** are (discrete) fuzzy subsets representing the linguistic values of V and

low then S is **high**" where D represents social dislocation as a result of a project and S represents associated "satisfaction." Fuzzy subsets, **low** and **high**, represent the linguistic values assumed by D and S, respectively. The method facilitates the inclusion of multiple antecedents combined either conjunctively or disjunctively, for example, "if V_1 is $\mathbf{A_1}$ AND V_2 is $\mathbf{A_2}$ then U is **B**" and "if V_1 is $\mathbf{A_1}$ OR V_2 is $\mathbf{A_2}$ then U is **B**," respectively.

This article presents a method for environmental evaluation based on *fuzzy relational equations* which relate two of the linguistic variables characterizing project assessment—"performance" (with respect to a criterion, impact, or factor) and "importance" (of a criterion, impact, or factor). Linguistic variables assume values such as "superior," "above average," "average," *etc.*, for performance and "critical," "important," "not important," *etc.*, for importance. The method is applicable to "soft" project assessment situations where precise quantitative data is unavailable for either the outcomes of projects with respect to factors (performance) or for the importance of factors selected for discriminating between projects.

FUZZY SETS AND FUZZY RELATIONS

A fuzzy set or, more precisely, *fuzzy subset,* is a set whose elements belong to the set in varying degrees. More formally a fuzzy subset **A** defined on X (a base set of objects denoted generically by x) is represented as $\mathbf{A} = \{A(x)|x, x \in X\}$ where A(x) is called the *membership value* or *grade of membership* [3-6]. For example, the fuzzy subset, **large_city**, (in the context of Australia, 1996) defined on base set $X = \{x_1\ x_2, x_3\ x_4\ x_5, x_6)$ = {Adelaide, Brisbane, Gold Coast, Melbourne, Perth, Sydney} might be expressed as **large_city** = {0.29|Adelaide, 0.39|Brisbane, 0.10|Gold Coast, 0.87|Melbourne, 0.33|Perth, 1|Sydney}. The power of fuzzy subsets lies in their ability to represent linguistic variables rather than quantitative variables [1-3].

The notion of *relation* is well known in mathematics and forms the basis for modeling complex systems such as environmental systems. Commonly a relation is a crisp, two-valued partition (related or unrelated) representing the presence or absence of an association, interaction, or interconnectedness between the elements of two or more sets [5]. A (*binary*) relation between two crisp sets is a subset of the *Cartesian product* $X \times Y$ denoted $R \subseteq X \times Y$. A crisp relation can be defined as $(R(x,y) = 1$, iff $(x,y) \in R$ and $R(x,y) = 0$, otherwise, where (x,y) is called a *tuple,* in particular, a *2-tuple.*

A *fuzzy relation* allows gradual transition between "related" and "unrelated," that is, for varying strengths of association, interaction, or interconnectedness between elements. A fuzzy relation is a fuzzy subset defined on the Cartesian product of crisp sets $X \times Y$ where 2-tuples (x,y) may have varying *degrees of membership* within the relation. In other words, for a fuzzy relation, $R(x,y) \in [0,1]$. For example, let $X = \{x_1, x_2, x_3)$ = {Sydney, Brisbane, Perth} and

$Y = \{ y_1, y_2, y_3) = \{$Melbourne, Alice Springs, Brisbane$\}$, then the relation "very far" might be represented by the fuzzy relation

		Y		
		(y_1) Melbourne	(y_2) Alice Springs	(y_3) Brisbane
X	(x_1) Sydney	0.19	0.57	0.21
	(x_2) Brisbane	0.36	0.66	0.00
	(x_3) Perth	0.74	0.79	0.96

That is, for example, $R(x_1, y_3)$ = (R(Sydney,Brisbane) = 0.21 represents the degree to which the Cities of Sydney and Brisbane satisfy the relation "very far" in terms of distance from each other.

FUZZY RELATIONAL EQUATIONS

Consider two binary relations **P** defined on $X \times Y$ and **Q** defined on $Y \times Z$, with a common base set Y. Then the *max-min composition,* **P** o **Q**, produces a binary relation **R** on $X \times Z$ defined by

$$R(x,z) = [\mathbf{P} \circ \mathbf{Q}](x,z) = \vee_{y\in Y}[P(x,y) \wedge Q(y,z)]$$

for all $x\in X$, $z\in Z$ [5, 6, 7]. Let N_x, N_y, and N_z denote the cardinalities (number of elements) of base sets X, Y, and Z, respectively.

Fuzzy relational equations are associated with the concept of composition of binary relations, **R** = **P** o **Q**. When **P** and **Q** are given in the composition, **R** = **P** o **Q**, the problem of determining **R** is trivial. However, when **R** and **Q** are given, the problem of determining **P** is less trivial, as is the case of determining **Q** when **R** and **P** are given. A solution may or may not exist and may not be unique.

Suppose that $\mathbf{P} = \{P(y), y\in Y\}$ and $\mathbf{R} = \{R(z), z\in Y\}$ are fuzzy subsets and $\mathbf{Q} = \{Q(y,z)\ y\in Y, z\in Z\}$ is a fuzzy relation. Further, let $N_Y = 4$ and $N_z = 3$ (note that $N_x = 1$ since **P** and **R** are fuzzy subsets), and let

$$\mathbf{P} = \{0.2|y_1, 0.0|y_2, 0.8|y_3, 1.0|y_4\}$$

and

$$\begin{aligned}\mathbf{Q} = \{&0.3|(y_1,z_1)\ 0.5|(y_1,z_2), 0.2|(y_1,z_3),\\ &0.8|(y_2,z_1)\ 1.0|(y_2,z_2), 0.0|(y_2,z_3),\\ &0.7|(y_3,z_1)\ 0.0|(y_3,z_2), 0.5|(y_3,z_3),\\ &0.6|(y_4,z_1)\ 0.3|(y_4,z_2), 1.0|(y_4,z_3)\}\end{aligned}$$

be given. Then the composition is as follows

$$\mathbf{R} = \mathbf{P} \circ \mathbf{Q} = \{0.7|z_1, 0.3|z_2, 1.0|z_3\}$$

where $R(z) = \vee_{y\in Y}[P(y) \wedge Q(y,z)]$, $z\in Z$. For example, $R(z_1) = \vee_{y\in Y}[P(y) \wedge Q(y,z_1)] = [P(y_1) \wedge Q(y_1,z_1)] \vee [P(y_2) \wedge Q(y_2,z_1)] \vee [P(y_3) \wedge Q(y_3,z_1)] \vee [P(y_4) \wedge Q(y_4,z_1)] = [0.2 \wedge 0.3] \vee [0.0 \wedge 0.8] \vee [0.8 \wedge 0.7] \vee [1.0 \wedge 0.6] = 0.2 \vee 0.0 \vee 0.7 \vee 0.6 = 0.7$.

If **P** and **R** are given in the composition, $\mathbf{R} = \mathbf{P} \circ \mathbf{Q}$, and **Q** is unknown, then Sanchez [8] has shown that there exists a unique maximum, $\hat{\mathbf{Q}}$, (in the sense of containment or inclusion) satisfying $\mathbf{R} = \mathbf{P} \circ \mathbf{Q}$. Thus, whenever the solution set $S(\mathbf{P},\mathbf{R}) = \{\mathbf{Q}|\mathbf{R} = \mathbf{P} \circ \mathbf{Q}\}$ is not empty (i.e., $S(\mathbf{P},\mathbf{R}) \neq \emptyset$, where $\emptyset$ is the empty set), it always contains a unique maximum solution, $\hat{\mathbf{Q}} \in S(\mathbf{P},\mathbf{R})$. $\hat{\mathbf{Q}}$ is given by $\hat{\mathbf{Q}} = \mathbf{P}^{-1} \circ_\alpha \mathbf{R}$, where, in membership terms, $\hat{Q}(y,z) = P(y)\ \alpha\ R(z)$. Here, **a α b** is the *α-relative pseudocomplement* of a in b (*Gödelian implication* [5]), defined as,

$$\begin{aligned} a\ \alpha\ b &= 1 \text{ if } a \leq b \\ &= b \text{ if } a > b \end{aligned}$$

where $a,b \in [0,1]$. This is a measure of the relative degree of containment of one grade of membership (a) in another (b) (that is, $\alpha\ b = a \subset b$). When $\hat{\mathbf{Q}}$ determined in this way does not satisfy $\mathbf{R} = \mathbf{P} \circ \mathbf{Q}$, then $S(\mathbf{P},\mathbf{R}) = \emptyset$. Note that $\mathbf{Q} \subseteq \hat{\mathbf{Q}}$.

In the above example, if **P** and **R** are given in the composition, $\mathbf{R} = \mathbf{P} \circ \mathbf{Q}$, and **Q** is unknown, then $\hat{\mathbf{Q}} = \mathbf{P}^{-1} \circ_\alpha \mathbf{R}$ **where,**

$$\begin{aligned} \hat{\mathbf{Q}} = \{&1.0|(y_1,z_1)\ 1.0|(y_1,z_2),\ 1.0|(y_1,z_3), \\ &1.0|(y_2,z_1)\ 1.0|(y_2,z_2),\ 1.0|(y_2,z_3), \\ &0.7|(y_3,z_1)\ 0.3|(y_3,z_2),\ 1.0|(y_3,z_3), \\ &0.7|(y_4,z_1)\ 0.3|(y_4,z_2),\ 1.0|(y_4,z_3)\} \end{aligned}$$

For example,

$$\begin{aligned} \hat{Q}(y_1,z_1) &= P(y_1)\ \alpha\ R(z_1) \\ &= 0.2\ \alpha\ 0.7 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \hat{Q}(y_2,z_1) &= P(y_2),\ \alpha\ R(z_1) \\ &= 0\ \alpha\ 0.7 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \hat{Q}(y_3,z_1) &= P(y_3)\ \alpha\ R(z_1) \\ &= 0.8\ \alpha\ 0.7 \\ &= 0.7 \end{aligned}$$

It can be shown that $\mathbf{P} \circ \hat{\mathbf{Q}} \subseteq \mathbf{R}$ and that $\mathbf{Q} \subseteq \hat{\mathbf{Q}}$. Note that while the α-composition between *fuzzy subsets,* **P** defined on base set Y, and **R**, defined on base set Z, is $P(y)\ \alpha\ R(z)$, $y \in Y, z \in Z$, the α-composition between *fuzzy relations* say, **P**, defined on $X \times Y$, and **R**, defined on $X \times Z$, is $\hat{\mathbf{Q}} = \mathbf{P}^{-1} \circ_\alpha \mathbf{R}$, or in membership terms, $\hat{Q}(y,z) = \wedge_{x \in X}[P(y,x)\ \alpha\ R(x,z)]$, $y \in Y, z \in Z$, where $P(y,x) = P(x,y)^{-1}$. Here, **Q** is unknown in the composition of *fuzzy relations,* $\mathbf{R} = \mathbf{P} \circ \mathbf{Q}$.

The α-composition between a *fuzzy subset,* **R**, and a *fuzzy relation,* **Q**, in the composition, $\mathbf{R} = \mathbf{P} \circ \mathbf{Q}$, (**P** unknown) is $\hat{\mathbf{P}} = (\mathbf{Q} \circ_\alpha \mathbf{R}^{-1})^{-1}$ or in membership terms, $\hat{P}(y) = \wedge_{z \in Z}[Q(y,z)\ \alpha\ R(z)]$. In this latter respect, assume that **R**, defined on Z, and **Q**, defined on $Y \times Z$, are given and **P** is unknown in the composition, $\mathbf{R} = \mathbf{P} \circ \mathbf{Q}$. Then, $\hat{\mathbf{P}} = (\mathbf{Q} \circ_\alpha \mathbf{R}^{-1})^{-1}$ where, in membership terms, $\hat{P}(y) = \wedge_{z \in Z}\{Q(y,z)\ \alpha\ R(z)\}$. $\hat{\mathbf{P}}$ is the largest **P** satisfying $\mathbf{R} = \mathbf{P} \circ \mathbf{Q}$. Thus, $\hat{\mathbf{P}} = \{0.3|y_1, 0.3|y_2, 1|y_3\ 1|y_4\}$ where, for example,

$$\begin{aligned}
\hat{P}(y_1) &= [Q(y_1,z_1)\ \alpha\ R(z_1)] \wedge [Q(y_1,z_2)\ \alpha\ R(z_2)] \wedge [Q(y_1,z_3)\ \alpha\ R(z_3)] \\
&= [0.3\ \alpha\ 0.7] \wedge [0.5\ \alpha\ 0.3] \wedge [0.2\ \alpha\ 1] \\
&= 1 \wedge 0.3 \wedge 1 \\
&= 0.3
\end{aligned}$$

$$\begin{aligned}
\hat{P}(y_2) &= [Q(y_2,z_1)\ \alpha\ R(z_1)] \wedge [Q(y_2,z_2)\ \alpha\ R(z_2)] \wedge [Q(y_2,z_3)\ \alpha\ R(z_3)] \\
&= [0.8\ \alpha\ 0.7] \wedge [1\ \alpha\ 0.3] \wedge [0\ \alpha\ 1] \\
&= 0.7 \wedge 0.3 \wedge 1 \\
&= 0.3
\end{aligned}$$

Again, note that $\hat{\mathbf{P}} \circ \mathbf{Q} \subseteq \mathbf{R}$ and that $\mathbf{P} \subseteq \hat{\mathbf{P}}$.

Generalized connectives for fuzzy relational equations have been proposed (for example [10, 11]). Pedrycz [10] proposes a generalized connective based on the fuzzy union and fuzzy intersection [12]. The *generalized intersection* operation, $\mathbf{A} \cap_p \mathbf{B}$, is defined as $A(x) \wedge_p B(x) = 1 - (1 \wedge \{(1 - A(x))^p + (1 - B(x))^p\}^{1/p})$ for $p \geq 1$, $x \in X$ and is a monotonically non-decreasing function of p; that is, if $p' < p''$, then $\mathbf{A} \cap_{p'} \mathbf{B} \leq \mathbf{A} \cap_{p''} \mathbf{B}$. When $p = \infty$, $A(x) \wedge_p B(x) = A(x) \wedge B(x)$ (*logical product*) and when $p = 1$, $A(x) \wedge_p B(x) = 0 \vee [A(x) + B(x) - 1]$ (*bounded product*). Thus, the parameter p is inversely related to the strength of the "and" (the lower the value of p, the stronger the "and"). The generalized composition is $\mathbf{P} \circ_p \mathbf{Q} = \mathbf{R}$, or in membership terms, $R(x,z) = [P \circ_p Q](x,z) = \sup_{y \in Y}[P(x,y) \wedge_p Q(y,z)]$. The resolution of the fuzzy relational equation, $\mathbf{P} \circ_p \mathbf{Q} = \mathbf{R}$, is given **P** and **R**, find **Q** (or, given **Q** and **R**, find **P**). A τ-operator is defined as $a\ \tau\ b = 1 - \{(1 - b)^p - (1 - a)^p\}^{1/p}$ if $a \geq b$, and $a\ \tau\ b = 1$, if $a < b$ ($a,b \in [0,1]$). The τ-operator is an example of a t-norm [5, 11]. Generalized connectives are not pursued in the example given below.

SYSTEMS OF FUZZY RELATIONAL EQUATIONS

Systems of fuzzy relational equations may be represented

$$\begin{aligned} R_1 &= P_1 \circ Q \\ R_2 &= P_2 \circ Q \\ &\vdots \\ R_m &= P_m \circ Q \end{aligned}$$

or, more concisely, as $R_\xi = P_\xi \circ Q$ ($\xi = 1,...,m$). A solution, if it exists (i.e., $\cap_{\xi=1,m} S(P_\xi, R_\xi) \neq \emptyset$), is given by $\hat{Q} \in \cap_{\xi=1,m} S(P_\xi R_\xi) = \cap_{\xi=1,m}(P_\xi^{-1} \circ_\alpha R_\xi)$ [9, 13].

FUZZY RELATIONAL EQUATIONS IN ENVIRONMENTAL PROJECT EVALUATION

Wilhelm and Parsaei outline a method for the use of linguistic variables to support the phased implementation of a computer integrated manufacturing (CIM) strategy [14]. This method involved two linguistic variables "importance" and "capability." These two linguistic variables allow the analyst to specify the importance associated with each of a set of goals common to all enabling technologies, and the capability of each technology to meet the strategic CIM goals of the organization.

In the context of environmental project evaluation, each project is defined in terms of a number of bio-physical and socioeconomic criteria or factors. Factors are commonly assumed to vary in terms of salience or importance. Thus, assume two linguistic variables—"importance" (ξ) and "performance" (η). Importance (ξ) is defined by primary values defined on base set $Z = \{z_1, z_2, ..., z_{11}\} = \{0.0, 0.1, 0.2, ..., 1\}$ as follows

	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
critical	0	0	0	0	0	0	0	0.05	0.15	0.8	1
important	0	0.1	0.25	0.75	0.9	1	0.9	0.75	0.25	0.1	0
unimportant	1	0.8	0.4	0.2	0.05	0	0	0	0	0	0

That is, for example, "important" is represented by a fuzzy subset, **important** = $\{0|z_1, 0.1|z_2, 0.25|z_3, 0.75|z_4, 0.9|z_5, 1|z_6, 0.9|z_7, 0.75|z_8, 0.25|z_9\ 0.1|z_{10}, 0|z_{11}\}$. Performance ($\eta$) is defined by primary values defined on base set $Y = \{y_1, y_2, ..., y_{11}\} = \{0.0, 0.1, 0.2, ..., 1\}$ as follows

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
superior	0	0	0	0	0	0	0	0.1	0.3	0.9	1
average	0	0.05	0.1	0.35	0.8	1	0.8	0.35	0.1	0.05	0
poor	0	0.8	0.4	0.2	0	0	0	0	0	0	0

That is, for example "average" is represented by a fuzzy subset, **average** = $\{0|y_1, 0.05|y_2, 0.1|y_3, 0.35|y_4, 0.8|y_5, 1|y_6, 0.8|y_7, 0.35|y_8, 0.1|y_9, 0.5|y_{10}, 0|y_{11}\}$. Thus, $N_Y = N_Z = 11$.

Given primary linguistic values, secondary linguistic values may be defined as **indeed_critical, more_or_less_critical, very_important, more_or_less important**, and **not_important** for "importance" and **indeed_superior, more_or_less superior, above_average, below_average**, and **very_poor** for "performance." These are defined as **indeed_critical** = int(**critical**), **more_or_less_critical** = **critical**$^{1/2}$, **very_important** = **important**^{2}, **more_or_less_important** = **important**$^{1/2}$, **not_important** = ¬**important** = $\{(1 - \text{important}(y))|y\}$, **indeed_superior** = int(**superior**), **more_or_less_superior** = **superior**$^{1/2}$, **above_average** = $\{0|y, y < 0.5, (1 - \text{average}(y))|y, y \geq 0.5\}$, **below_average** = $\{(1 - \text{average}(y))|y, y < 0.5, 0|y, y \geq 0.5\}$ and **very_poor** = **poor**^{2}. The *intensification* function is defined as

$$\begin{aligned} \text{int}(A(x)) &= 2A(x)^2 && \text{if } A(x) < 0.5 \\ &= 1 - 2(1 - A(x))^2 && \text{if } A(x) \geq 0.5 \end{aligned}$$

The intensification operator has the effect of increasing high membership values (≥ 0.5) and decreasing low membership values (< 0.5). The linguistic values for "importance" and "performance" are therefore

	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
indeed_critical	0	0	0	0	0	0	0	0.01	0.05	0.92	1
more_or_less_critical	0	0	0	0	0	0	0	0.22	0.39	0.89	1
very_important	0	0.01	0.06	0.56	0.81	1	0.81	0.56	0.06	0.01	0
more_or_less_important	0	0.32	0.50	0.87	0.95	1	0.95	0.87	0.50	0.32	0
not_important	1	0.90	0.75	0.25	0.10	0	0.10	0.25	0.75	0.90	1

and

	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
indeed_superior	0	0	0	0	0	0	0	0.02	0.18	0.98	1
more_or_less_superior	0	0	0	0	0	0	0	0.32	0.55	0.95	1
above_average	0	0	0	0	0	0	0.20	0.65	0.90	0.95	1
below_average	1	0.95	0.90	0.65	0.20	0	0	0	0	0	0
very_poor	1	0.64	0.16	0.04	0	0	0	0	0	0	0

These fuzzy subsets (based on [14]) are shown in Figure 1 and Figure 2 respectively.

Assume a fuzzy relational equation, $\eta^j_k = \zeta_k \circ \psi_{kj}$, where η^j_k represents "performance" of project j with respect to factor k (defined on base set Y),

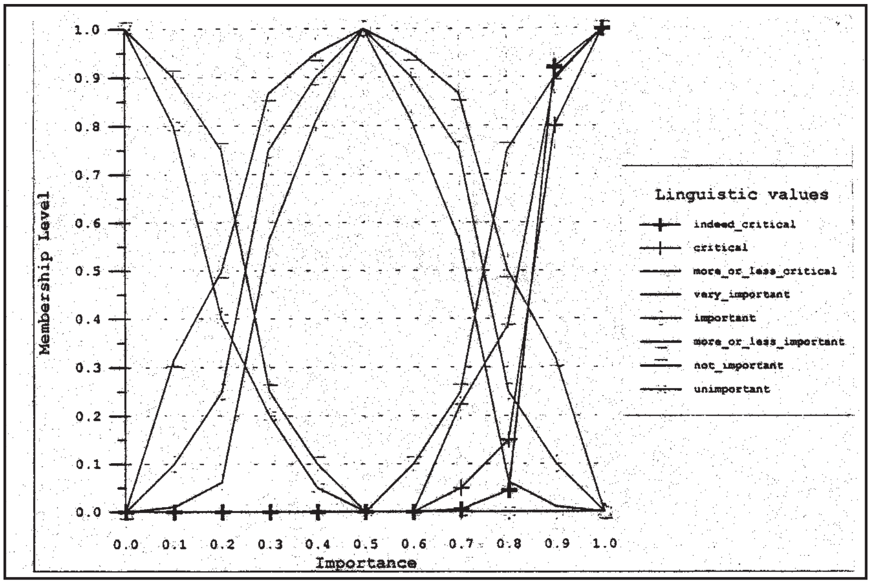

Figure 1. Linguistic values of Importance.

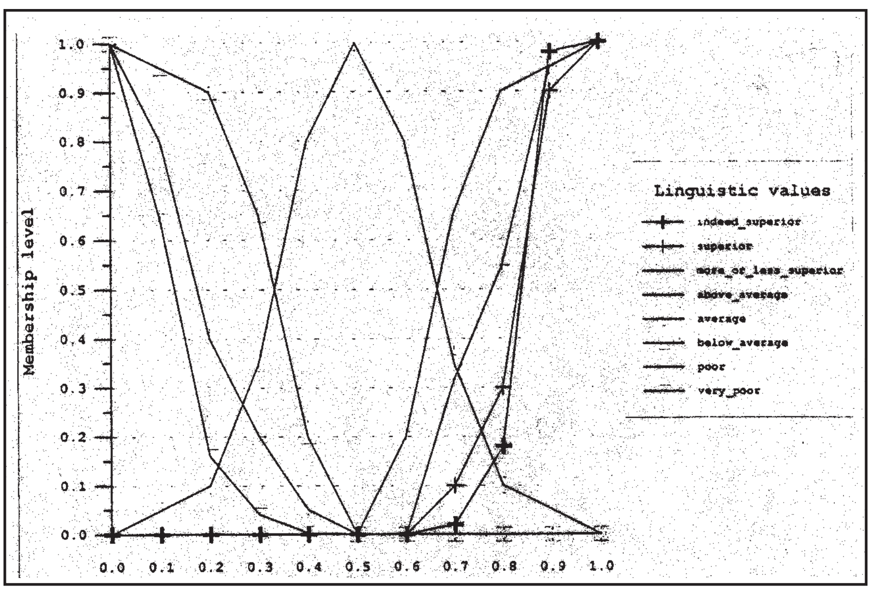

Figure 2. Linguistic values of Performance.

ζ_k represents "importance" of factor k (defined on base set Z). Thus, $\eta^j_k = \{\eta^j_k(y_1)|y_1, \eta^j_k(y_2)|y_2,...,\eta^j_k(y_{11})|y_{11}\}$ and $\zeta_k = \{\zeta_k(z_1)|z_1, \zeta_k(z_2)|z_2,..., \zeta_k(z_{11})|z_{11}\}$. ψ_{kj} represents a linguistic assessment of the performance of project j with respect to factor k (defined on base set Z × Y), that is, in membership terms, $\psi_{kj} = \{\psi_{kj}(z_1,y_1)|(z_1,y_1), \psi_{kj}(z_1,y_2)|(z_1,y_2),..., \psi_{kj}(z_1,y_{11})|(z_1,y_{11}),..., \psi_{kj}(z_{11},y_1)|(z_{11},y_1), \psi_{kj}(z_{11},y_2)|(z_{11},y_2),..., \psi_{kj}(z_{11},y_{11})|(z_{11},y_{11})\}$. In membership terms, the min-max composition is $\eta^j_k(y) = \vee_{z\varepsilon Z}[\zeta_k(z) \wedge \psi_{kj}(z,y)]$.

Given an assessment of importance and performance for a factor-project combination, then the largest ψ_{kj} satisfying $\eta^j_k = \zeta_k \circ \psi_{kj}$ is given by $\hat{\psi}_{kj} = \zeta_k^{-1} \circ_\alpha \eta^j_k$, or, in membership terms, $\hat{\psi}_{kj}(z,y) = \zeta_k(z) \,\alpha\, \eta^j_k(y)$. Note that $\zeta_k \circ \hat{\psi}_{kj} \subseteq \eta^j_k$ and $\psi_{kj} \subseteq \hat{\psi}_{kj}$.

The intersection of fuzzy relations, $\hat{\psi}_{kj}$, across factors/impacts is then $R_j = \cap_{k=1,m}\hat{\psi}_{kj} = \cap_{k=1,m}(\zeta_k^{-1} \circ_\alpha \eta^j_k)$, or, in membership terms, $R_j(z,y) = \wedge_{k=1,m} \hat{\psi}_{kj}(z,y) = \wedge_{k=1,m}(\zeta_k(z) \,\alpha\, \eta^j_k(y))$. Again, assume that the importance of satisfying all factors/impacts is "indeed critical." Then, in the relational equation, $\eta^{j*} = \zeta^* \circ R_j$, solve for η^{j*} (the overall performance of project j), where ζ^* = **indeed_critical**. In membership terms, this is $\eta^{j*}(y) = \vee_{z\varepsilon Z} [\zeta^*(z) \wedge R_j(z,y)]$. In this method, both ζ^* and R_j are known, so that their max-min composition yields η^{j*}.

Note that Wilhelm and Parsaei [14] use a composition equivalent to $\zeta_k = \eta^j_k \circ \phi_{jk}$, where ζ_k represents "importance" of factor k, η^j_k represents "performance" (i.e., "capability") with respect to factor k, and $\phi_{jk} = \{\phi_{jk}(y,z)|(\langle y,z\rangle), y\in Y, z\varepsilon Z\}$, defined on base set Y × Z, represents a linguistic assessment of the performance of project j with respect to factor k. In membership terms, the max-min composition is $\zeta_k(z) = \vee_{y\varepsilon Y}[\eta^j_k(y) \wedge \phi_{jk}(y,z)]$. Thus $\hat{\phi}_{jk} = (\eta^j_k)^{-1} \circ_\alpha \zeta_k$ is the largest ϕ_{jk} satisfying $\zeta_k = \eta^j_k \circ \phi_{jk}$, or, in membership terms, $\hat{\phi}_{jk}(y,z) = \eta^j_k(y) \,\alpha\, \zeta_k(z)$. Again, $Z_j = \cap_{k=1,m}\hat{\phi}_{jk}$, or, $Z_j(y,z) = \wedge_{k=1,m}\hat{\phi}_{jk}(y,z)$. However, in this case, the relational equation, $\zeta_k = \eta^j \circ Z_j$, is solved for η^j (the performance of project j) as $\hat{\eta}^j = (Z_j \circ_\alpha \zeta^{*-1})^{-1}$ where ζ^* = **indeed_critical**. In membership terms, this is $\hat{\eta}^j(y) = \wedge_{z\varepsilon Z}(Z_j(y,z) \,\alpha\, \zeta^*(z))$. $\hat{\eta}_j$ is the largest η^j which satisfies $\zeta^* = \eta^j \circ Z_j$.

Wilhelm and Parsaei [14] adopt the *Hamming distance* to identify the fuzzy subset, $\hat{\eta}^j$, representing the closest to the fuzzy subset, **indeed_superior**. The Hamming distance between two fuzzy subsets $\mathbf{A} = \Sigma A(x)|x$ and $\mathbf{B} = \Sigma B(x)|x$ is defined as $H(\mathbf{A},\mathbf{B}) = (1/n)\Sigma_{i=1,n}|A(x_i) - B(x_i)|$ where n is the cardinality of X. Thus, $H(\hat{\eta}^j$, **indeed_superior**$) = (1/11)\Sigma_{i=1,11}|\hat{\eta}^j(y_i) -$ indeed_superior$(y_i)|$.

Methods for "defuzzifying" each $\hat{\eta}_j$ may also be used to rank order projects. Defuzzification methods include the *point value* (PV) method [15, 16] and the *center of area* (COA), *center of gravity* (COG) or *centroid* method [5]. The point value of a fuzzy subset $\mathbf{A} = \Sigma A(x_i)|x_i$ is given by

$$F(\mathbf{A}) = (1/\alpha_{max}) \int_0^{\alpha_{max}} M(A_\alpha)\, d\alpha$$

where α_{max} is the maximum grade of membership of A and A_α is the alpha level set of A. An alpha level set is a crisp set $A_\alpha = \{x_i | A(x_i) \geq \alpha\}$. $M(A_\alpha)$ is the mean value of A_α (see also [3]).

The COA method is as follows

$$COA(A) = \Sigma_{i=1,n} A(x_i)x_i / \Sigma_{i=1,n} A(x_i)$$

where $A = \Sigma A(x)|x$ and $X = \{x_1, x_2, ..., x_n\}$ [15].

EXAMPLE

Consider the assessment of three transport projects (route alignments) relative to seven factors—capital and maintenance cost, travel-time savings, wildlife impact, air quality impact, land-use impact, noise impact, and social dislocation. Suppose that the three projects are assessed against each of the impacts/factors as shown in Table 1.

Thus, project j = 1 performs strongly in terms of economic/engineering factors, project j = 2 performs strongly in terms of environmental factors, and project j = 3 performs somewhat variably across economic/engineering and environmental factors.

Table 1. Performance

Factor	Project 1	Project 2	Project 3	Importance
Capital/ Maintenance Cost	indeed superior	poor	average	important
Travel-Time Savings	indeed superior	below average	superior	very important
Wildlife Impact	very poor	indeed superior	above average	indeed critical
Air Quality Impact	below average	superior	above average	important
Land-Use Impact	below average	indeed superior	below average	important
Noise Impact	very poor	indeed superior	superior	critical
Social Dislocation	very poor	superior	poor	important

Here, $\eta^j_k = \zeta_k \circ \psi_{kj}$, where ζ_k represents "importance" of factor k (k = 1,2,...,7), η^j_k represents "performance" of project j (j = 1,2,3) with respect to factor k, and ψ_{kj} represents a linguistic assessment of the performance of project j with respect to factor k. Given an assessment of importance and performance for a factor-project combination, then $\hat{\psi}_{kj} = \zeta_k{-}1 \ o_\alpha \ \eta^j_k$ represents the largest ψ_{kj} such that $\eta^j_k = \zeta_k \circ \psi_{kj}$.

For example, for project j = 1 and factor k = 1 (Capital/Maintenance Costs), $\hat{\psi}_{11} = \zeta_1{-}1 \ o_\alpha \ \eta^1_1$ is given by **important**$^{-1}$ o_α **indeed_superior** = $\{0|z_1, 0.1|z_2, 0.25|z_3, 0.75|z_4, 0.9|z_5, 1|z_6, 0.9|z_7, 0.75|z_8, 0.25|z_9, 0.1|z_{10}, 0|z_{11}\}^{-1}$ o_α $\{0|y_1, 0|y_2, 0|y_3, 0|y_4, 0|y_5\ 0|y_6, 0|y_7, 0.02|y_8, 0.18|y_9, 0.98|y_{10}, 1|y_{11}\}$. Then,

		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
z_1	0	1	1	1	1	1	1	1	1	1	1	1
z_2	0.1	0	0	0	0	0	0	0	0.02	1	1	1
z_3	0.2	0	0	0	0	0	0	0	0.02	0.18	1	1
z_4	0.3	0	0	0	0	0	0	0	0.02	0.18	1	1
z_5	0.4	0	0	0	0	0	0	0	0.02	0.18	1	1
z_6	0.5	0	0	0	0	0	0	0	0.02	0.18	0.98	1
z_7	0.6	0	0	0	0	0	0	0	0.02	0.18	1	1
z_8	0.7	0	0	0	0	0	0	0	0.02	0.18	1	1
z_9	0.8	0	0	0	0	0	0	0	0.02	0.18	1	1
z_{10}	0.9	0	0	0	0	0	0	0	0.02	0.18	1	1
z_{11}	1	1	1	1	1	1	1	1	1	1	1	1

The intersection of fuzzy relations, $\hat{\psi}_{kj}$, across factors/impacts is given as $\mathbf{R}_j = \cap_{k=1,7} \hat{\psi}_{kj}$. $\mathbf{R}_1$, for example, is given as

		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
z_1	0	1	1	1	1	1	1	1	1	1	1	1
z_2	0.1	0	0	0	0	0	0	0	0	0	0	0
z_3	0.2	0	0	0	0	0	0	0	0	0	0	0
z_4	0.3	0	0	0	0	0	0	0	0	0	0	0
z_5	0.4	0	0	0	0	0	0	0	0	0	0	0
z_6	0.5	0	0	0	0	0	0	0	0	0	0	0
z_7	0.6	0	0	0	0	0	0	0	0	0	0	0
z_8	0.7	0	0	0	0	0	0	0	0	0	0	0
z_9	0.8	0	0	0	0	0	0	0	0	0	0	0
z_{10}	0.9	0	0	0	0	0	0	0	0	0	0	0
z_{11}	1	1	0.64	0.16	0.04	0	0	0	0	0	0	0

Again, assume that the importance of satisfying all factors/impacts is "indeed critical." Then, in the relational equation, $\eta^{j*} = \zeta^{*} \text{ o } \mathbf{R_j}$, solve for η^{j*} (the performance of project j), where ζ^{*} = **indeed_critical**. The fuzzy subsets η^{j*} (j = 1,2,3) are as follows

	y_1 0.0	y_2 0.1	y_3 0.2	y_4 0.3	y_5 0.4	y_6 0.5	y_7 0.6	y_8 0.7	y_9 0.8	y_{10} 0.9	y_{11} 1.0
Project 1	1	0.64	0.16	0.04	0	0	0	0	0	0	0
Project 2	0	0	0	0	0	0	0	0.02	0.18	0.98	1
Project 3	0	0	0	0	0	0	0	0.1	0.3	0.9	1

and each may then be defuzzified using the point value or the center of area. Thus $PV_1 = 0.042$, $PV_2 = 0.941$, and $PV_3 = 0.935$ and $COA_1 = 0.059$, $COA_2 = 0.936$, and $COA_3 = 0.922$. Project 2 is "best" in terms of overall performance.

The original method of Wilhelm and Parsaei [14] yields the following results

	y_1 0.0	y_2 0.1	y_3 0.2	y_4 0.3	y_5 0.4	y_6 0.5	y_7 0.6	y_8 0.7	y_9 0.8	y_{10} 0.9	y_{11} 1.0
Project 1	1	1	1	0.045	0.005	0	0	0	0	0	0
Project 2	0	0	0	0	0	0	0	0.045	1	1	1
Project 3	0	0	0	0	0	0	1	1	1	1	1

Thus $PV_1 = 0.103$, $PV_2 = 0.898$, and $PV_3 = 0.8$ and $COA_1 = 0.103$, $COA_2 = 0.897$, and $COA_3 = 0.8$. Again, project 2 is "best" in terms of overall performance. However, the previous solution appears to provide a finer discrimination between projects.

CONCLUSION

A simple example of the use of fuzzy relational equations in environmental project evaluation has been presented based on a variation of a method proposed by Wilhelm and Parsaei [14] in the context computer integrated manufacturing strategy. The method is applicable where only soft data, that is, linguistic expressions of performance with respect to impacts/factors, are available.

Further research is necessary to explore the merits of fuzzy relational equations in project evaluation, the implications of different fuzzy subset representations of linguistic variables, the merit of generalizations to the context of other t-norm based compositions, etc. Possibilities for combining linguistic expressions of project performance in terms of some impacts/factors with quantitative expression of performance on other impacts/factors might also be usefully explored.

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