On Flow through Renal Tubule in case of Periodic Radial Velocity Component

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ABSTRACT
The problem of fluid motion in a renal tubule, in contrast to ordinary flow through cylinder with the impermeable wall, is complicated by the existence of radial velocity generated by reabsorption process. It has been experienced that due to some abnormality the porosity of wall may get hampered in turn radial velocity may become periodic. The main objective of this paper is to read the behaviour of blood flow through renal tubule in case of periodic radial velocity component. It is the fact that the radial component of velocity of blood is responsible to balance the salt concentration between dialyzate and blood stream. Our analysis shows that the blood flow pattern changes and flow is not smooth.

Key words: Renal tubule, flow through tube, Bessel functions.

NOMENCLATURE
R – radius of the narrow tube
r – radial position
v_r – fluid velocity in radial direction
v_z – fluid velocity in axial direction
μ – viscosity of viscous incompressible fluid
p – pressure
ϕ – general function of radial distance z
ψ – stream function
λ – frequency parameter
$f(r)$ – unknown function
$g(r)$ – unknown function

1. INTRODUCTION
One of the most useful and important organs of a human body is kidney. There are two kidneys in the human body, located on either side of the spinal column in the posterior of the abdomen. The kidneys perform a wide range of functions including, regulation of water, regulation of blood pressure, regulation of ionic plasma concentrations and elimination of metabolic waste products such as urea, uric acid, creatinine, etc.. Blood is supplied to each kidney via the renal artery through a slit as the concave medial surface called the renal hilus. The functional unit of the kidney is the nephron or renal tubule. Each nephron is individually capable of regulating the volume of water and concentration of soluble substances by filtering the blood, reabsorbing useful components, and excreting the rest. Each human kidney contains approximately one million nephrons. A nephron consists of two functionally different parts, the glomerulus and the renal tubule. The glomerulus creates an ultra filtrate of body by sieving out blood cells and large plasma proteins. The tubule lacks a mechanism to reabsorb these. If the kidneys deliver this filtrate for excretion, the body loses many valuable materials, including water at a rate faster than the one at which they can be supplied by synthesis or feeding. The rest of the

nephron recovers these valuable materials and returns them to the blood. Thus about 80% of the filtrate is reabsorbed in the proximal tubule, and the remaining about 95% is further reabsorbed by the end of the collecting ducts. The bulk phase of reabsorption occurs in the proximal tubule. All the proteins, glucose, and amino acids from the glomerular filtrate are reabsorbed along with approximately 65% of the water.

Modeling of the vascular system began with Euler in 1775. Since 1965 there have been several models proposed to describe 1-D or 2-D flows in non-compliant tubules or especially in the renal tubule with its reabsorbing endothelium. Each model begins with a simplified set of the Navier-Stokes and continuity equations under the assumptions of an incompressible Newtonian or non-Newtonian fluid. In the non-compliant models this closes the system; in the compliant models an equation describing the behavior of the wall is necessary. The diversity in the models comes from the variety of equations of motion used for the tubule wall and from the reabsorption patterns [1]. Macey [2] was the first to study the mathematical modeling of the flow in proximal tubule. He formulated the problem as the flow of an incompressible viscous fluid through a circular tube with linear rate of reabsorption at the wall. Kelman [3] noted that the bulk flow in the proximal tube decays exponentially with axial distance. Later in 1965, Macey [4] used this condition and solve the equations of motions to find pressure drop. Marshal and Trowbridge [5] and Palatt et al. [6] used physical conditions existing at the permeable wall instead of prescribing the flux/radial velocity at the wall.

In all the above analysis the renal tubule is assumed as cylindrical tube of uniform cross section. Radhakrishnamacharya et al. [7] made an attempt to understand the flow through the renal tubule by the hydro-dynamical aspect of an incompressible viscous fluid in a circular tube of varying cross section with reabsorption at the wall. Krishnaprasad and Chandra [8] analyzed the flow in rigid tubes of slowly varying cross section with absorbing wall. Chaturani and Ranganathan [9] considered fluid flow through a diverging /converging tube with variable wall permeability. Muthu and Tesfahun [10, 11] made an attempt to understand the flow through renal tubule by studying the hydro-dynamical aspect of an incompressible viscous fluid in rigid tube of slowly varying cross section with reabsorption at the wall.

As explained by Mazumdar [12], the main function of kidneys is to maintain the chemical balance of blood by excreting the waste products such as urea, creatine and uric acid in the blood stream. An important diffusion process takes place in renal tubules in kidneys, in the form of diffusion of urea through semi-permeable membrane of dialyser. Also, an important fluid flow takes place in the functional unit of the kidney known as nephron or renal tubule. Each kidney has about a million of these tubules. Here, reabsorption of water and other low molecular weight substances takes place along the walls of the renal tubule, and both radial and axial flows are present.

Assuming that during the balancing process between blood stream and dialyser, due to some unknown reasons, the pores of semi-permeable wall may get blocked and in turn the radial velocity component becomes periodic, we solve the flow problem here analytically and see the effect of periodicity of radial component on axial velocity component.

2. MATHEMATICAL FORMULATION

Consider a long narrow tube having uniform cross-section of radius R. The axis of tube is z-axis and the radial axis is perpendicular to the axis of tube. The model is similar to the flow model through kidney as shown in Figure 1 and it has been governed by the following equations:

$$\frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \tag{1}$$

$$\frac{1}{\mu}\frac{\partial p}{\partial r} = \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \tag{2}$$

$$\frac{1}{\mu}\frac{\partial p}{\partial z} = \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \tag{3}$$

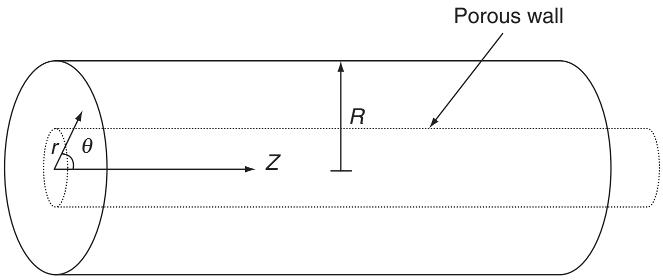

Figure 1. Systematic sketch of renal tubule.

where, μ is viscosity of viscous incompressible fluid, p is pressure, v_r is the fluid velocity in radial direction, and v_z is axial component of velocity. The relevant boundary conditions considered for model are:

$$\begin{aligned} &r = 0, v_r = 0, \ \frac{\partial v_z}{\partial r} = 0 \\ &r = R, \ v_r = \phi(z), v_z = 0. \end{aligned} \tag{4}$$

Eliminating the pressure gradient between (1) and (2) and defining stream function $\psi(r, z)$ by the relation

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial z}, \ \ v_z = -\frac{1}{r}\frac{\partial \psi}{\partial r} \tag{5}$$

we get

$$\nabla^4 \psi = 0 \ \text{ or } \ \nabla^2(\nabla^2 \psi) = 0 \tag{6}$$

where, $\nabla^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$. In order to solve the equation (6), we try the solution of the form

$$\psi(r, z) = f(r)\cos\lambda z + g(r). \tag{7}$$

Putting $\left(D^2 - \frac{1}{r}D - \lambda^2\right) f = F,$ we get the following ordinary differential equation of order two.

$$F'' - \frac{1}{r}F' - \lambda^2 F = 0. \tag{8}$$

Referring Handbook of Mathematical Functions by Abramowitz and Stegun [13], the solution of general Bessel's equation

$$r^2 F'' + \left[(1-2A)r - 2Br^2\right]F' + \left[C^2 D^2 r^{2C} + Br^2 - B(1-2A)r + A^2 - C^2 n^2\right]F = 0 \tag{9}$$

is

$$F(r) = r^A e^{Br}\left[C_1 I_n(Dr^C) + C_2 K_n(Dr^C)\right]. \tag{10}$$

On comparing the equation (8) in the form

$$r^2F'' - rF' - (\lambda r)^2 F = 0$$

with the equation (10), we have $A = 1$, $B = 0$, $C = 1$, $B = i\lambda$, $D = i\lambda$ and $n = 1$. Here we see that D is imaginary. In 1985, Jung Y. Yoo and Daniel D. Josef [14] encountered with the same situation. Following the same method, we carry forward our calculations.

Therefore, the complete solution of bi-harmonic equation becomes

$$f(r) = r\left[C_1 I_1(\lambda r) + C_2 K_1(\lambda r) + r(C_3 I_2(\lambda r) + C_4 K_2(\lambda r))\right] \tag{11}$$

where I_1, I_2 are modified Bessel's functions of first kind and K_1, K_2 are of second kind. To satisfy boundary conditions at $r = 0$, we must have $C_2 = 0 = C_4$. Therefore, the solution is

$$f(r) = r\left[C_1 I_1(\lambda r) + rC_3 I_2(\lambda r)\right]. \tag{12}$$

Using the equation (7) in equation (6) and applying the method of separation of variables, we get

$$\left(f^{iv} - \frac{2}{r}f''' + \frac{3}{r^2}f'' - \frac{3}{r^3}f' - 2\lambda^2 f'' + \frac{2}{r}f'\lambda^2 + \lambda^4 f\right)\cos\lambda z + \left(g^{iv} - \frac{2}{r}g''' + \frac{3}{r^2}g'' - \frac{3}{r^3}g'\right) = 0 \tag{13}$$

Equating the coefficient of $\cos\lambda z$ and the term independent of $\cos\lambda z$ to zero, we have

$$\left(f^{iv} - \frac{2}{r}f''' + \frac{3}{r^2}f'' - \frac{3}{r^3}f' - 2\lambda^2 f'' + \frac{2}{r}f'\lambda^2 + \lambda^4 f\right) = 0$$

and

$$\left(g^{iv} - \frac{2}{r}g''' + \frac{3}{r^2}g'' - \frac{3}{r^3}g'\right) = 0$$

$$\Rightarrow \qquad \left(D^2 - \frac{1}{r}D - \lambda^2\right)^2 f = 0, \text{ and } \left(D^2 - \frac{1}{r}D\right)^2 g = 0.$$

The solution of non-linear differential equation

$$\left(D^2 - \frac{1}{r}D\right)^2 g = 0. \tag{14}$$

is

$$g(r) = \frac{A_1}{4}r^4 + \frac{r^2}{2}\left\{A_2(\ln r - \frac{1}{2}) + A_3\right\} + A_4 \tag{15}$$

Using the equations (12) and (15), we have the stream function as follows

$$\psi(r,z) = r\left[C_1 I_1(\lambda r) + r\,C_3 I_2(\lambda r)\right]\cos\lambda z + \frac{A_1}{4}r^4 + \frac{r^2}{2}\left\{A_2(\ln r - \frac{1}{2}) + A_3\right\} + A_4. \tag{16}$$

Hence, the velocity components are

$$v_r = -\lambda\ \left[C_1 I_1(\lambda r) + r\,C_3 I_2(\lambda r)\right]\sin\lambda z \text{ , and}$$

$$v_z = -\frac{1}{r}\left(f'(r)\cos\lambda z + g'(r)\right).$$

Applying the boundary conditions at $r = 0$, we get

$$\frac{\partial v_z}{\partial r} = -\left[-\frac{1}{r^2}(f'(r)\cos\lambda z + g'(r)) + \frac{1}{r}\left(f''(r)\cos\lambda z + g''(r)\right)\right]$$

$$= -\left[-\frac{1}{r^2}[\left\{C_1\left(\lambda r I_0(\lambda r)\right) + C_3\left(\lambda r^2 I_1(\lambda r)\right)\right\}\cos\lambda z + \frac{A_1}{4}r^3 + A_2 r\ln r + A_3 r]\right]$$

$$+\frac{1}{r}\left[[C_1\left\{\lambda(I_0(\lambda r) + r\lambda I_1(\lambda r))\right\} + C_3\lambda\left\{rI_1(\lambda r) + r^2\lambda I_0(\lambda r)\right\}]\cos\lambda z + \frac{3}{4}A_1 r^3 + A_2(1+\ln r) + A_3\right]$$

$$= [C_1\lambda\frac{I_0(\lambda r)}{r} + C_3\lambda I_1(\lambda r)]\cos\lambda z + \frac{A_1}{4}r + A_2\frac{\ln r}{r} + \frac{A_3}{r}]$$

$$+[C_1(\lambda\frac{I_0(\lambda r)}{r} + \lambda I_1(\lambda r)) + C_3\lambda(I_1(\lambda r) + \lambda r I_0(\lambda r))]\cos\lambda z + \frac{3}{4}A_1 r^2 + A_2\frac{1+\ln r}{r} + \frac{A_3}{r}$$

where A_2, A_3 and C_1 must be zero (= 0). Substituting A_2, A_3 and C_1 in v_r and v_z, then applying boundary condition at $r = R$, we get

$$C_3 = -\frac{\cos\lambda z}{\lambda R I_2(\lambda R)\sin\lambda z}\text{ , and } A_1 = -\frac{4I_1(\lambda R)\cos^2(\lambda z)}{I_2(\lambda R)}$$

Hence, the radial and axial components of velocity are

$$v_r = \frac{r}{R}\left(\frac{I_2(\lambda r)}{I_2(\lambda R)}\right)\cos(\lambda z) \text{ and } v_z = \left(\frac{I_1(\lambda r)}{I_2(\lambda R)} - \frac{I_1(\lambda R)}{I_2(\lambda R)}\left(\frac{r}{R}\right)\right)\left(\frac{r}{R}\right)\frac{\cos^2(\lambda z)}{\sin\lambda z}. \tag{17}$$

3. RESULTS AND DISCUSSION

The mathematical model, where the radial component of velocity has been assumed periodic, has been solved for the velocity components v_r and v_z completely satisfying the prescribed boundary conditions. To emphasis our findings, we draw the graphs by taking the random values of parameters. Reading the graphs in Fig. 2, we see that the behaviour of radial velocity component v_r is periodic. The frequency of oscillations increases as λ increases. Due to reabsorption process, both radial and axial components of the velocity of flow exist. As radial velocity component is periodic, so the diffusion process of salt may get affected. Hence, an abnormality or impurity which leads to blockage of the pores of wall membrane due to which the radial velocity becomes periodic is very much adverse to the health.

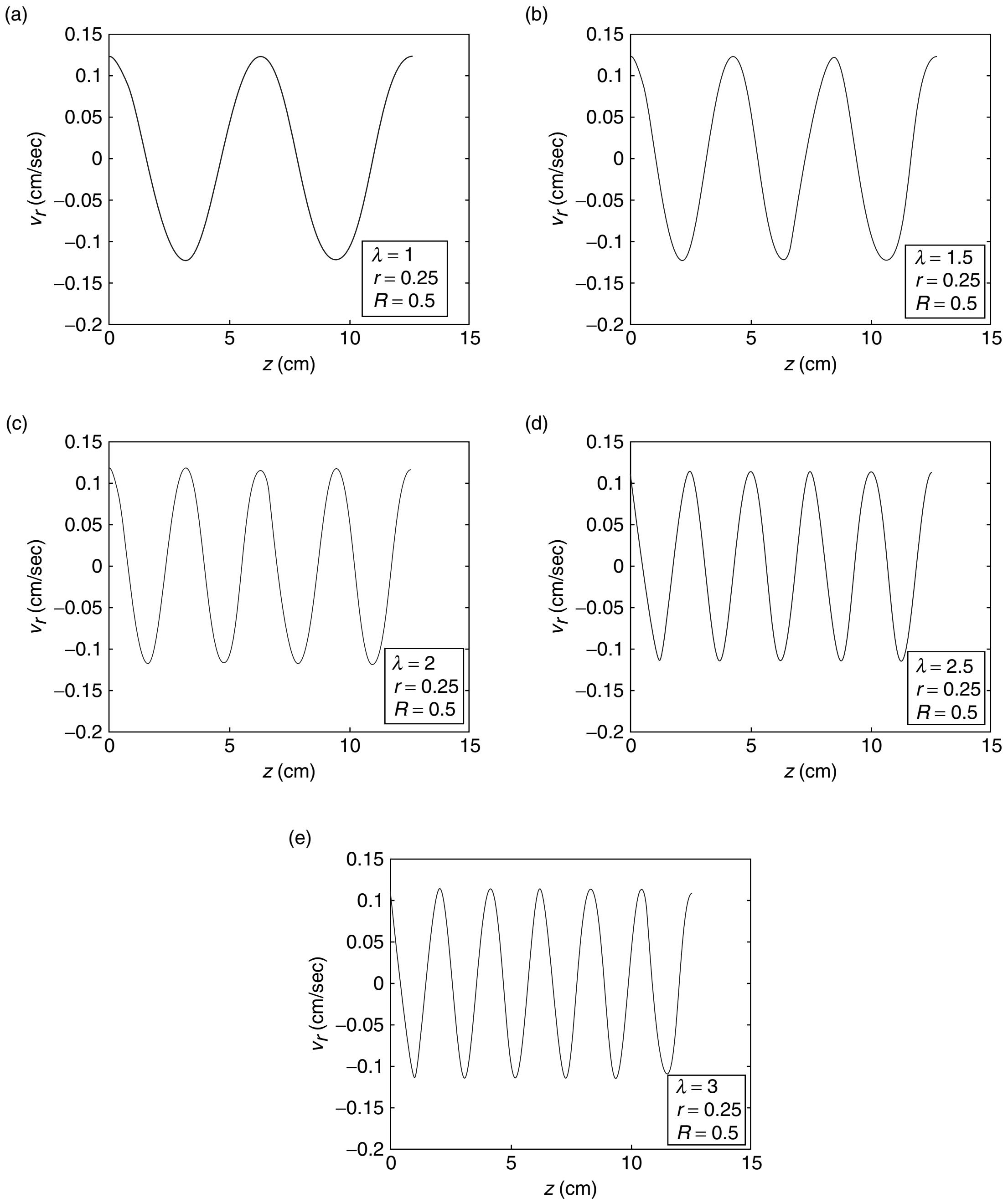

Figure 2. Radial velocity for different values of frequency parameter λ when R = 0.5 and r = 0.25.

Looking at the expression for v_z, we find that there are the places (points) where the requirement of continuity has not been met, i.e. $\sin \lambda z = 0 \Rightarrow z = 0$, and $z = \dfrac{n\pi}{\lambda}, n \in \{0,1,2,,.\}$. These points are singularities in dynamic region. Peak and trough of axial velocity exist due the involvement of cosine and sine functions. On the basis of fluid mechanics, we may say that the diffusion process gets disturbed, in turn Creatinine, Urea in blood, etc. may be found. Naturally the blood composition will not be normal, hence a person becomes ill. In Fig. 3, we see this fact and the frequency of disturbance increases as the frequency parameter λ increases.

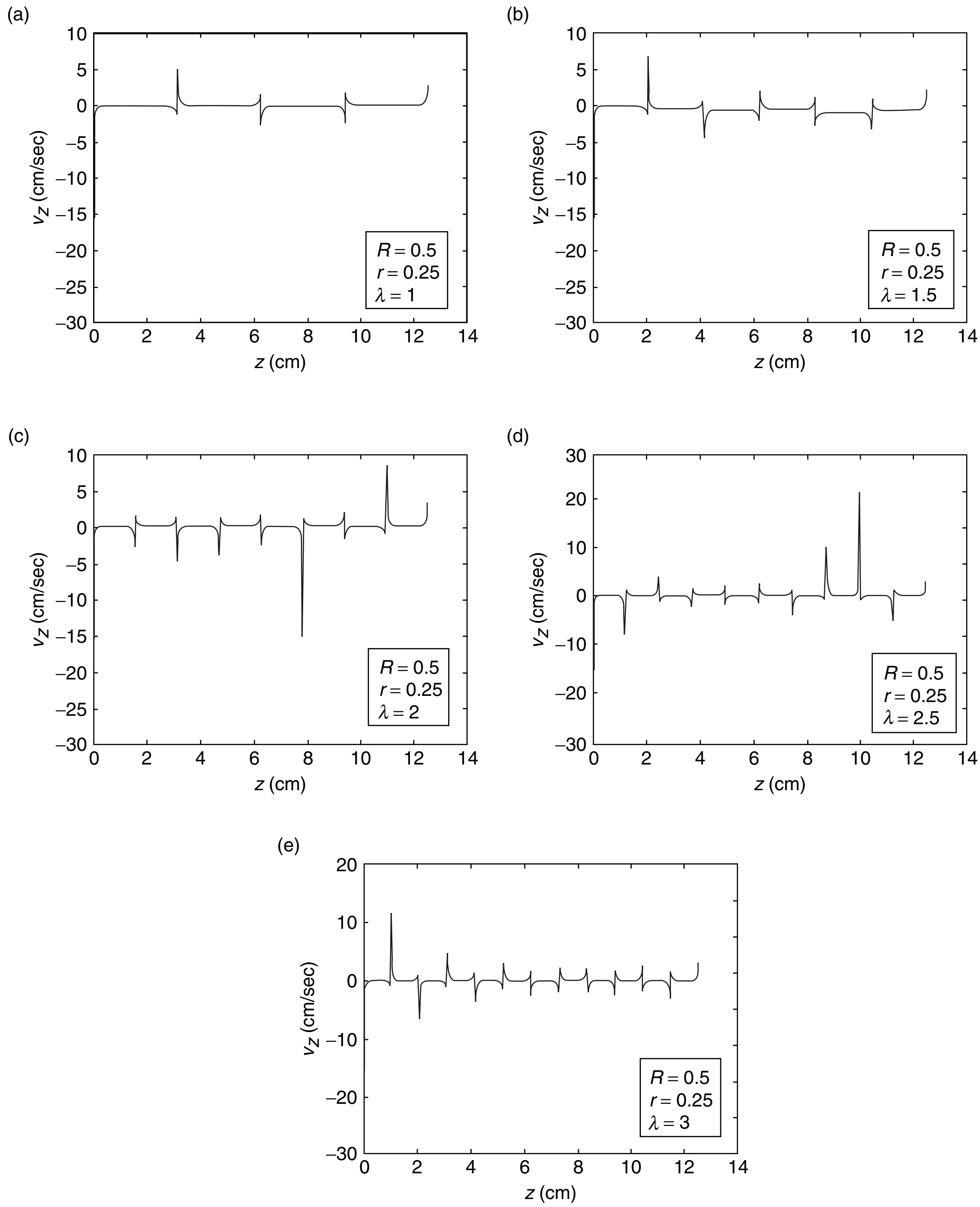

Figure 3. Axial velocity for different values of frequency parameter λ when R = 0.5 and r = 0.25.

4. CONCLUSION

In the course of analysis of flow through renal tubule in case of periodic radial velocity component, we came across the following conclusions:

(i) Sickness to the human body comes in various forms. In this paper, we assume that the porous wall of renal tubule gets disturbed so that the radial component of velocity becomes periodic. The radial component of velocity is mainly responsible to balance the concentration of different salts in the blood stream. Figures 2*a* through 2*e* exhibit the behaviour of radial velocity and the effect of frequency parameter λ.

(ii) The periodic nature of radial component of velocity affects the blood flow stream, i.e. axial component of velocity shown in Figures 3*a* through 3*e* is not smooth and uniform. There is some disturbance in blood stream which is not a good for normal health.

Hence, the disturbance in radial component is bad for health and it may lead to the serious sickness in kidney functions.

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